UC Berkeley Math 254A
Problem Set 3

October 10, 2014

Last Updated: October 10, 2014. Please let me know if you find any typos.

We will discuss these questions in class on **Wed. Oct. 22th.**
Written solutions are due on **Fri. Oct. 24nd.**

1 Prime ideals vs. Discrete valuations vs. DVR's

Let $L|K$ be a finite extension of fields. Let $\mathcal{O}_K$ be a Dedekind domain with fraction field K and let $\mathcal{O}_L$ be the integral closure of $\mathcal{O}_K$ in L. Discuss the relationship between the following three concepts:

1. An extension $\mathfrak{P}|\mathfrak{p}$ of prime ideals of $\mathcal{O}_L|\mathcal{O}_K$ – i.e. $\mathfrak{P}$ is a prime ideal of $\mathcal{O}_L$ such that $\mathfrak{P} \cap \mathcal{O}_K = \mathfrak{p}$.
2. An extension $B|A$ of DVR's of $L|K$ – i.e. a DVR A of K and a DVR B of L such that $B \cap K = A$.
3. An extension $w|v$ of discrete valuations of $L|K$ – i.e. a discrete valuation v of K and a discrete valuation w of L such that $w|_K = v$.

Here are some questions to guide your discussion:

1. Given an extension of prime ideals, produce an associated extension of DVR's and an associated extension of discrete valuations.
2. Given an extension of DVR's, produce an associated extension of prime ideals, and an associated extension of discrete valuations.
3. Given an extension of discrete valuations, produce an associated extension of prime ideals, and an associated extension of DVR's.
4. Prove that the definition of ramification and inertia degrees for $\mathfrak{P}|\mathfrak{p}$ resp. $B|A$ resp. $w|v$ all agree under these identifications.
5. In class, we defined the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ to be $\mathcal{O}_K/\mathfrak{p}$ (since $\mathfrak{p}$ is maximal). Give a compatible definition of $\kappa(A)$ for A a DVR, and of $\kappa(v)$ for v a discrete valuation.
6. In the case where $L|K$ is Galois, we defined the decomposition/inertia subgroups of $\mathrm{Gal}(L|K)$ associated to $\mathfrak{P}|\mathfrak{p}$. Give a definition of decomposition/inertia groups for an extension of DVR's resp. discrete valuations which is compatible with the definition for $\mathfrak{P}|\mathfrak{p}$ via the identifications above.

2 Localizations and valuation rings

1. Suppose that A is a Noetherian domain. Prove that A is a Dedekind domain if and only if for all prime ideals $0 \neq \mathfrak{p}$ of A, the localization $A_{\mathfrak{p}}$ is a DVR.

2. We say that a local ring $(A, \mathfrak{m})$ is **dominated** by another local ring $(B, \mathfrak{m}')$ if $A \subset B$ and $\mathfrak{m}' \cap A = \mathfrak{m}$. Let K be a field and consider the collection S of all local subrings of K whose fraction field is K; consider S as a partially ordered set with respect to the domination relation defined above. Prove that a local ring $(A, \mathfrak{m})$ is a valuation ring of K (as defined on HW2) if and only if $(A, \mathfrak{m})$ is a maximal element of S.

3. Suppose that $\mathcal{O}_K$ is a Dedekind domain with fraction field K. Let A be a DVR whose fraction field is K. We say that A is **centered** on $\mathcal{O}$ if $\mathcal{O} \subset A$. Let $C_{\mathcal{O}}$ be the set of all DVR's of K which are centered on $\mathcal{O}$.

   (a) Prove that $A \in C_{\mathcal{O}}$ if and only if $A = \mathcal{O}_{\mathfrak{p}}$ for some non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$.

   (b) Prove that $\mathcal{O} = \bigcap_{A \in C_{\mathcal{O}}} A$.

   (c) Prove that $\mathcal{O}^{\times} = \bigcap_{A \in C_{\mathcal{O}}} A^{\times}$.

   (d) Give a description of (b) and (c) in terms of the discrete valuations $v_{\mathfrak{p}}$ associated to prime ideals $\mathfrak{p}$ of $\mathcal{O}$.

3 Decomposition theory

1. Suppose that $M|L|K$ is a tower of finite extensions of fields, and that $\mathcal{O}_K$ is a Dedekind domain of K with integral closure $\mathcal{O}_L$ resp. $\mathcal{O}_M$ in L resp. M. Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_K$, and let $\mathfrak{P}$ be a prime of $\mathcal{O}_M$ such that $\mathfrak{P}|\mathfrak{p}$; furthermore, let $\mathfrak{P}' = \mathfrak{P} \cap L$ so that $\mathfrak{P}|\mathfrak{P}'$ and $\mathfrak{P}'|\mathfrak{p}$. Prove that $e(\mathfrak{P}|\mathfrak{p}) = e(\mathfrak{P}|\mathfrak{P}') \cdot e(\mathfrak{P}'|\mathfrak{p})$ and $f(\mathfrak{P}|\mathfrak{p}) = f(\mathfrak{P}|\mathfrak{P}') \cdot f(\mathfrak{P}'|\mathfrak{p})$.

2. Let $L|K$ be a finite separable extension of fields and let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_K$. Let N be the Galois closure of $L|K$. Prove or disprove and salvage the following statements:

   (a) $\mathfrak{p}$ is totally split in L if and only if $\mathfrak{p}$ is totally split in N.

   (b) $\mathfrak{p}$ is ramified in L if and only if $\mathfrak{p}$ is ramified in N.

   (c) $\mathfrak{p}$ is inert in L if and only if $\mathfrak{p}$ is inert in N.

3. Let $L|K$ and $L'|K$ be two finite separable extension and let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_K$. Let $N = LL'$ be the compositum of L and L'. Prove or disprove and salvage each of the following statements:

   (a) $\mathfrak{p}$ is totally split in L and in L' then $\mathfrak{p}$ is totally split in N.

   (b) $\mathfrak{p}$ is unramified in L and in L' then $\mathfrak{p}$ is unramified in N.

   (c) $\mathfrak{p}$ is inert in L and in L' then $\mathfrak{p}$ is inert in N.

4. Let $L|K$ be a finite Galois extension of number fields. Consider the set S of prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$, such that, for some (hence all) $\mathfrak{P} \in X_{\mathfrak{p}}(L)$ one has $Z_{\mathfrak{P}|\mathfrak{p}} = 1$. Prove that S has density 1 if and only if $L = K$.

4 Conductors of integral primitive elements

1. Let $\mathcal{O}_K$ be a Dedekind domain with fraction field K, let L be a finite separable extension of K, and let $\mathcal{O}_L$ denote the intergral closure of $\mathcal{O}_K$ in L. Let $\theta \in \mathcal{O}_L$ be a primitive element. Suppose that $\mathfrak{p}$ is a non-zero prime ideal of $\mathcal{O}_K$ which is prime to the conductor of $\mathcal{O}_K[\theta]$. Prove that $(\mathcal{O}_L)_\mathfrak{p} = (\mathcal{O}_K)_\mathfrak{p}[\theta]$.

2. Suppose that $L|K$ is a finite extension of number fields, and let $\theta \in \mathcal{O}_L$ be a primitive element of $L|K$. Prove that the index $[\mathcal{O}_L : \mathcal{O}_K[\theta]]$ is finite, and that the conductor of $\mathcal{O}_K[\theta]$ divides $[\mathcal{O}_L : \mathcal{O}_K[\theta]] \cdot \mathcal{O}_L$.

3. Let $L|K$ be a finite separable extension, and let $\mathfrak{a}$ be an integral ideal of $\mathcal{O}_L$. Prove that there exists a $\theta \in \mathcal{O}_L$ such that $L = K(\theta)$ and $\mathfrak{a}$ is coprime to the conductor of $\mathcal{O}_K[\theta]$.