UC BERKELEY MATH 254A
PROBLEM SET 1

Last Updated: September 10, 2014. Please let me know if you find any typos.

We will discuss these questions in class on Monday, Sept. 22nd.

(1) Questions about integral closure:

(a) Show that $\mathbb{C}[x, y]/(x^2 - y^3)$ is a domain which is not normal.
(b) Show that A is normal if and only if $A[t]$ is normal.
(c) Let D be a square-free integer. Compute an integral basis for the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{D})$. What is the discriminant of K?
(d) Find an integral basis for $\mathbb{Q}(\sqrt[3]{2})$. What is the discriminant of $\mathbb{Q}(\sqrt[3]{2})$?
(e) Let $A \subset B$ be an integral extension of rings. Let $\mathfrak{p}$ be a non-trivial prime-ideal of B. Prove that $\mathfrak{p} \cap A$ is non-trivial, and that $\mathfrak{p}$ is maximal if and only if $\mathfrak{p} \cap A$ is maximal.

(2) Questions about Norms and Traces:

(a) Give a proof of the following proposition proved in class:

Proposition. *Let $M|L|K$ be a tower of finite seperable extensions of fields. Then $N_{M|K} = N_{L|K} \circ N_{M|L}$ and $\mathrm{Tr}_{M|K} = \mathrm{Tr}_{L|K} \circ \mathrm{Tr}_{M|L}$.*

(b) Suppose that $L|K$ is a finite separable extension. Prove that $\mathrm{Tr}_{L|K}$ is non-zero.
(c) Let $L|K$ be a finite extension of fields. Prove that

$$\mathrm{Tr}_{L|K}(\alpha) = [L : K(\alpha)] \cdot \mathrm{Tr}_{K(\alpha)|K}(\alpha),$$

and that

$$N_{L|K}(\alpha) = N_{K(\alpha)|K}(\alpha)^{[L:K(\alpha)]}.$$

(d) Suppose that $L|K$ is a finite extension which is not separable. Prove that $\mathrm{Tr}_{L|K} = 0$. Deduce that if $M|L|K$ is a tower of finite (possibly non-separable) extensions of fields, then $\mathrm{Tr}_{M|K} = \mathrm{Tr}_{L|K} \circ \mathrm{Tr}_{M|L}$. Can you prove similar statements about Norms?
(e) Let K be a number field and let $a \in \mathcal{O}_K$ be given. Prove that

$$|N_{K|\mathbb{Q}}(a)| = [\mathcal{O}_K : (a)].$$

(3) Recall the following proposition proved in class:

Proposition. *Let K be a number field and let $\mathfrak{a} \subset \mathfrak{a}'$ be two finitely-generated $\mathcal{O}_K$-submodules of K. Then $d(\mathfrak{a}) = [\mathfrak{a}' : \mathfrak{a}]^2 \cdot d(\mathfrak{a}')$.*

Try to formulate and prove an analogue of this proposition for K a global function field. Hint: What is the correct analogue of $[\mathfrak{a}' : \mathfrak{a}]$? What is $[\mathfrak{a}' : \mathfrak{a}] \cdot (\mathfrak{a}'/\mathfrak{a})$?

(4) Questions about Dedekind domains:

(a) Prove that a Dedekind domain with finitely many prime ideals is a principal ideal domain. Hint: if $\mathfrak{a} = \mathfrak{p}_1^{v_1} \cdots \mathfrak{p}_r^{v_r}$ is a non-zero ideal, choose elements $\pi_i \in \mathfrak{p}_i \smallsetminus \mathfrak{p}_i^2$. Now consider $(\pi_i^{v_i} \mod \mathfrak{p}_i^{v_i+1})_i \in \prod_i A/\mathfrak{p}_i^{v_i+1}$ and apply the Chinese remainder theorem.

(b) Suppose that A is a domain in which every non-zero ideal admits a unique factorization as a product of prime ideals. Prove that A must be a Dedekind domain.

(5) The absolute Norm: Let K be a number field and let $\mathfrak{a}$,$\mathfrak{b}$ be ideals of $\mathcal{O}_K$. Define the **absolute norm** of $\mathfrak{a}$, denoted $\mathrm{N}(\mathfrak{a})$, to be $[\mathcal{O}_K : \mathfrak{a}]$ (compare with (2)(e) above)

(a) Prove that $\mathrm{N}(\mathfrak{a}\mathfrak{b}) = \mathrm{N}(\mathfrak{a}) \cdot \mathrm{N}(\mathfrak{b})$. Hint: write $\mathfrak{a} = \mathfrak{p}_1^{v_1} \cdots \mathfrak{p}_r^{v_r}$, with $\mathfrak{p}_i$ distinct primes, and prove that $\mathrm{N}(\mathfrak{a}) = \prod_i \mathrm{N}(\mathfrak{p}_i)^{v_i}$.

(b) Let K be a global function field. Use your definition of $[\mathcal{O}_K : \mathfrak{a}]$ from number (3) above in the case where K is a global function field to define $\mathrm{N}(\mathfrak{a})$ for an ideal $\mathfrak{a}$ of $\mathcal{O}_K$. Can you prove that $\mathrm{N}(\mathfrak{a}\mathfrak{b}) = \mathrm{N}(\mathfrak{a})\,\mathrm{N}(\mathfrak{b})$?