THE YONEDA LEMMA
MATH 250B

ADAM TOPAZ

1. The Yoneda Lemma

The Yoneda Lemma is a result in abstract category theory. Essentially, it states that objects in a category $\mathcal{C}$ can be viewed (functorially) as presheaves on the category $\mathcal{C}$.

Before we state the main theorem, we introduce a bit of notation to make our lives easier. For a category $\mathcal{C}$ and an object $A \in \mathcal{C}$, we denote by h_A the functor $\mathrm{Hom}^{\mathcal{C}}(\bullet, A)$ (recall that this is a contravariant functor). Similarly, we denote the functor $\mathrm{Hom}^{\mathcal{C}}(A, \bullet)$ by h^A. The statement of the Yoneda Lemma (in contravariant form) is the following.

Theorem: Let $\mathcal{C}$ be a category. *If F is an arbitrary contravariant functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, then one has*

$$\mathrm{Hom}^{\mathrm{Fun}}(h_A, F) \cong F(A).$$

This isomorphism is functorial in A in the sense that one has an isomorphism of functors:

$$\mathrm{Hom}^{\mathrm{Fun}}(h_\bullet, F) \cong F(\bullet).$$

Before we prove the statement, let us show that $h_\bullet$ is actually a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$. Namely, suppose that $A \to B$ is a morphism in $\mathcal{C}$, we must show how to construct a natural transformation $h_f : h_A \to h_B$. Namely, for an object $C \in \mathcal{C}$, we need to associate a

$$h_f(C) : h_A(C) \to h_B(C).$$

This map is the only natural choice: one has $h_A(C) = \mathrm{Hom}^{\mathcal{C}}(C, A)$ and $h_B(C) = \mathrm{Hom}^{\mathcal{C}}(C, B)$. The map $h_f(C) : \mathrm{Hom}^{\mathcal{C}}(C, A) \to \mathrm{Hom}^{\mathcal{C}}(C, B)$ is defined to be the map $h^C(f)$ (post-composition with f). The fact that this makes $h_\bullet$ into a functor is left as an (easy) exercise. It is now clear that $\mathrm{Hom}^{\mathrm{Fu}}(h_\bullet, F)$ is actually a functor, so the statement of the theorem at least makes sense.

We now prove the Theorem. To do this, we must construct mutually inverse natural transformations $\mathrm{Hom}^{\mathrm{Fu}}(h_\bullet, F) \to F$ and $F \to \mathrm{Hom}^{\mathrm{Fu}}(h_\bullet, F)$.

First we look for a natural transformation $F \to \mathrm{Hom}^{\mathrm{Fu}}(h_\bullet, F)$. For $A \in \mathcal{C}$, consider the map

$$F(A) \to \mathrm{Hom}^{\mathrm{Fun}}(h_A, F)$$

which sends an element $a \in F(A)$ to the natural transformation

$$\eta_a : h_A \to F$$

whose associated morphism on an object B:

$$\eta_a(B) : \mathrm{Hom}^{\mathcal{C}}(B, A) \to F(B)$$

is given by

$$\eta_a(B)(f) = F(f)(a) \in F(B).$$

We leave it as an exercise to check that this is a natural transformation.

Conversely, we look for a natural transformation $\mathrm{Hom}^{\mathrm{Fun}}(h_\bullet, F) \to F$. For $A \in \mathcal{C}$, consider the map

$$\mathrm{Hom}^{\mathrm{Fun}}(h_A, F) \to F(A)$$

which sends a natural transformation $\eta \in \mathrm{Hom}^{\mathrm{Fun}}(h_A, F)$ to the element $\eta(A)(\mathbf{1}_A) \in F(A)$ which is defined as follows. Namely, $\eta(A)$ is a map $\mathrm{Hom}^{\mathcal{C}}(A, A) \to F(A)$, and $\mathrm{Hom}^{\mathcal{C}}(A, A)$ has a distinguished element $\mathbf{1}_A$; so the element of $F(A)$ associated to η is the image of this distinguished element under $\eta(A)$.

The proof that these two functors are mutually inverse to eachother essentially boils down to the following observation. A natural transformation $\eta \in \mathrm{Hom}^{\mathcal{C}}(h_A, F)$ is completely determined by $\eta(A)(\mathbf{1}_A) =: \xi_A$. For an element $f \in \mathrm{Hom}^{\mathcal{C}}(B, A)$, one has a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}^{\mathcal{C}}(A, A) \arrow[r, "\phi \mapsto \phi \circ f"] \arrow[d, "\eta(A)"'] & \mathrm{Hom}^{\mathcal{C}}(B, A) \arrow[d, "\eta(B)"] \\
F(A) \arrow[r, "F(f)"'] & F(B).
\end{tikzcd}$$

So we see that if $f \in \mathrm{Hom}^{\mathcal{C}}(B, A)$ is a morphism, one has $\eta(B)(f) \in F(B)$, and this element of $F(B)$ is precisely given by $F(f)(\xi_A)$, since $f = \mathbf{1}_A \circ f$. The details are left to the reader.

2. Yoneda Embedding Theorem

We saw above that the assignment

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$$

defined by $A \mapsto h_A$ is actually a (covariant) functor. The Yoneda Embedding Theorem is a special case of the Yoneda Lemma described above. It satates the followsing:

Theorem: *For all $A, B \in \mathcal{C}$, one has an isomorphism*

$$\mathrm{Hom}^{\mathrm{Fun}}(h_A, h_B) \cong \mathrm{Hom}^{\mathcal{C}}(A, B).$$

In particular, the functor $A \mapsto h_A$ from $\mathcal{C}$ to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ is fully faithful, and injective.

In other words, the functor $h_\bullet$ is a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$. The image of this functor is the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ which consists of representable presheaves, and $h_\bullet$ induces an isomorphism between $\mathcal{C}$ and this full subcategory.

This is a special case of the Yoneda Lemma above, if we simply take $F = h_B$ in the previous theorem. Indeed, then one has an isomorphism

$$\mathrm{Hom}^{\mathrm{Fun}}(h_A, h_B) \cong h_B(A) = \mathrm{Hom}^{\mathcal{C}}(A, B).$$

To see that the functor $h_\bullet$ is fully faithful, we simply note that the map on Hom-sets:

$$\mathrm{Hom}^{\mathcal{C}}(A, B) \to \mathrm{Hom}^{\mathcal{C}}(h_A, h_B)$$

is precisely the map which was defined in the course of the proof of the Yoneda Lemma above. In particular, this is a bijection. The fact that $h_\bullet$ is injective is trivial.

3. Covariant Versions

We can easily state a covariant version of the Yoneda Lemma.

Theorem: Let $\mathcal{C}$ be a category. *If F is an arbitrary covariant functor $\mathcal{C} \to \mathbf{Set}$, then one has*

$$\mathrm{Hom}^{\mathrm{Fun}}(h^A, F) \cong F(A).$$

This isomorphism is functorial in A in the sense that one has an isomorphism of functors:

$$\mathrm{Hom}^{\mathrm{Fun}}(h^\bullet, F) \cong F(\bullet).$$

In this case, the functor $h^\bullet$ is a contravariant functor from $\mathcal{C}$ to $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$. The analogous Yoneda Embedding theorem is essenitally the same:

Theorem: *For all $A, B \in \mathcal{C}$, one has an isomorphism*

$$\mathrm{Hom}^{\mathrm{Fun}}(h^A, h^B) \cong \mathrm{Hom}^{\mathcal{C}}(B, A).$$

In particular, the functor $A \mapsto h^A$ from $\mathcal{C}^{\mathrm{op}}$ to $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ *is fully faithful, and injective.*

The proofs of these theorems is essentially the same as the proof above, taking into account the direction of the arrows. In fact, the covariant version of the Yoneda Lemma follows immediately by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$ by duality (i.e. $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$).