UC BERKELEY MATH 250B
PROBLEM SET 6

Last Updated: March 20, 2015. Please let me know if you find any typos.

Solutions are due on **Thursday, April 2nd, 2015**.

As always, all rings are assumed to be commutative with identity.

1. Chain Conditions

(1) Let A be a ring and let $I_1, \ldots, I_m$ be ideals of A such that the following hold:
  (a) A/I_i is Noetherian for each $i = 1, \ldots, m$.
  (b) One has $(0) = I_1 \cdots I_m$.

  Prove that A is Noetherian.

(2) Let A be a local ring with maximal ideal $\mathfrak{m}$. Assume that $\mathfrak{m}$ is principal, and that $\bigcap_n \mathfrak{m}^n = (0)$. Prove that A is Noetherian, and that every ideal of A is a power of $\mathfrak{m}$.

(3) Let A be a Noetherian ring and let $\phi : A \to A$ be a surjective morphism. Prove that ϕ is also injective.

(4) Find an example of a ring A which shows that the "Noetherian" condition is required in the previous problem.

2. Spec of a Ring

(1) Prove the statement mentioned in class: $\operatorname{Spec} A$ is finite and discrete if and only if A is Artinian.

(2) Let M be an A-module, and let $f_1, \ldots, f_m \in A$ be such that $(f_1, \ldots, f_m) = A$. Prove the following:
  (a) Let $m \in M$ be given. If the image of m is zero in $M \otimes_A A_{f_i}$ for each i, then $m = 0$.
  (b) Let $m_i \in M \otimes_A A_{f_i}$, for $i = 1, \ldots, r$ be given. Assume that the image of m_i is the same as the image of m_j in $M \otimes_A A_{f_i f_j}$ for each i, j. Prove that there exists some $m \in M$ such that the image of m in $M \otimes_A A_{f_i}$ is m_i for each i.

  Hint: Use an argument which is similar to what we did in class on 03/19/2015.

3. Idempotents

Let A be a ring and let $e \in A$ be such that $e \neq 0, 1$ and such that e is idempotent, i.e. $e^2 = e$.

(1) Prove that $e' := 1 - e$ is also idempotent.

(2) Consider the closed sets of $\operatorname{Spec} A$ given by $V(e) = C_1$ and $V(1 - e) = C_2$. Prove that $C_1 \cap C_2 = \varnothing$, that $C_1, C_2 \neq \varnothing$, and that $\operatorname{Spec} A = C_1 \cup C_2$. Deduce that $\operatorname{Spec} A$ is disconnected.

(3) Prove that the canonical map $A \to A_e \times A_{e'}$ is an isomorphism.

Now suppose that B is another ring, and assume that $\operatorname{Spec} B$ is disconnected, i.e. there exist closed subsets C_1, C_2 of $\operatorname{Spec} B$ such that $C_1 \cup C_2 = \operatorname{Spec} B$, that $C_1, C_2 \neq \varnothing$, and that $C_1 \cap C_2 = \varnothing$.

(1) Prove that there exist ideals I and J of B such that $IJ \subset \sqrt{(0)}$ and such that $I + J = B$.
(2) Use the previous problem to prove that there exist $a \in I$ and $b \in J$ such that $a + b = 1$ and such that $ab = 0$.
(3) Deduce that $a^2 = a$ and that $b^2 = b$ (these play the role of e resp. e' from the previous problems).

4. Connected and Irreducible Components

Let A be a Noetherian ring.

(1) Use the problems from the previous section to prove that $\operatorname{Spec} A$ has finitely many connected components.
(2) Prove that $\operatorname{Spec} A$ is a union of *finitely many* irreducible closed subsets.
Hint: recall that any irreducible closed subset of $\operatorname{Spec} A$ is of the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Which primes correspond to the maximal irreducible subsets of $\operatorname{Spec} A$?
(3) Give examples which show that (1) and (2) above are false if the "Notherian" condition is dropped.