UC BERKELEY MATH 250B
PROBLEM SET 3

Last Updated: February 15, 2015. Please let me know if you find any typos.

Solutions are due on **Tuesday, Feb. 24th, 2015**.

1. Derived Functors and Ext

(1) Let F be a functor from the category of R-modules to the category of abelian groups. Prove that F is additive if and only if $F(A \oplus B) \cong F(A) \oplus F(B)$ functorially in A and B.

(2) Let F be an additive functor from the category of R-modules to the category of abelian groups. Prove that $L_i F$ and $R^i F$ are additive for all i. (Hint: Use the previous problem.)

(3) Prove that P is projective if and only if $\mathrm{Ext}^i_R(P, A) = 0$ for all A. Prove that I is injective if and only if $\mathrm{Ext}^i_R(A, I) = 0$ for all A.

(4) Compute the following ext groups, for all $i \geq 0$, all abelian groups A, and all torsion abelian groups B:

$$\mathrm{Ext}^i_{\mathbb{Z}}(B, \mathbb{Z}),\ \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n, A).$$

Optional: try to compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.

(5) An R-module A is said to have **projective dimension** $\leq m$ if $\mathrm{Ext}^i_R(A, B) = 0$ for all $i > m$ and all B. Prove that the following are equivalent:
   (a) A has projective dimension $\leq m$.
   (b) $\mathrm{Ext}^{m+1}_R(A, B) = 0$ for all B.
   (c) There exists a porjective resolution $(P_*)_{*\geq 0}$ of M such that $P_i = 0$ for $i > m$.
   (d) In every projective resolution $(P_*)_{*\geq 0}$ of A, the image of $P_m \to P_{m-1}$ is projective (if $m = 0$, we put $P_{-1} := A$).

2. Group Cohomology and Group Rings

(1) Let F denote the free group on a set S. Prove that the augmentation ideal $I(F) := \ker(\mathbb{Z}[F] \to \mathbb{Z})$ is a free $\mathbb{Z}[F]$-module. Deduce that $\mathrm{H}^*(F, A) = 0$ for all $* \geq 2$ and all F-modules A, i.e. F has *cohomological dimension* ≤ 1.

(2) Let G be a group and let H be a subgroup of G. Prove that $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module.

(3) Let $\mathcal{C}$ be the category whose objects are pairs (G, A) where G is a group and A is a G-module. A morphism $(G, A) \to (G', A')$ in $\mathcal{C}$ consists of a homomorphism of groups $G' \to G$ and a morhism $A \to A'$ of G'-modules, where A is endowed with the structure of a G'-module via the map $G' \to G$. Let $C^*(G, A)$ denote the complex of inhomogeneous cochains of G with values in A. Prove that the assignment $(G, A) \mapsto C^*(G, A)$ is a covariant functor from $\mathcal{C}$ to the category of complexes of abelian groups. Deduce that the assignment $(G, A) \mapsto \mathrm{H}^*(G, A)$ is a covariant functor from $\mathcal{C}$ to the category of abelian groups.

3. Induced Modules

Let G be a group and H a subgroup of G. Let A be an H-module. The **induced** module of A is a G-module whose underlying abelian group is the group

$$\mathrm{Functions}_H(G, A) =: M_G^H(A)$$

of *functions* $f : G \to A$ such that $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$. **Note:** some texts use the term **co-induced** module for $M_G^H(A)$.

(1) Prove that $M_G^H(A)$ is a left G-module, where the action is defined by $gf(g') = f(g'g)$ for $g, g' \in G$.
(2) Prove that $M_G^H(A) \cong \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A)$ as G-modules, where $g \in G$ acts on $f \in \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], A)$ by $(gf)(\alpha) = f(\alpha \cdot g)$.
(3) Prove that the map $M_G^H(A) \to A$ defined by $f \mapsto f(1)$ induces a morphism $(G, M_G^H(A)) \to (H, A)$ in the category $\mathcal{C}$ from (3) above, where the corredponsing map $H \to G$ is the canonical inclusion. Deduce that there is a canonical map $\mathrm{H}^*(G, M_G^H(A)) \to \mathrm{H}^*(H, A)$, which is functorial in A.
(4) Prove that the map $\mathrm{H}^*(G, M_G^H(A)) \to \mathrm{H}^*(H, A)$ is an isomorphism (Hint: use (2) from this section, combined with (2) from the previous section, and use projective resolutions to calculate cohomology.)
(5) Recall that for an abelian group X, the group $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X)$ has the natural structure of a G-module. Prove that $\mathrm{H}^*(G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X)) = 0$ for all $* \geq 1$ and all abelian groups X.