INTRODUCTIO TO HOMOLOGICAL ALGEBRA
MATH 250B

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Mostly everything we say in this note can be stated in the context of an *abelian category*. The notion of an abelian category is one where we can do standard constructions dealing with abelian groups, such as:

(1) One has a zero object.
(2) Taking finite products, finite direct sums (coproducts).
(3) Hom-sets can be endowed with the structure of an abelian group, and composition is distributive with respect to addition of morphisms.
(4) Any morphism has a kernel and cokernel and image.
(5) One has a notion of a subgroup, and all inclusions have cokernels (i.e. we can form quotients).

The prototype of an abelian category is the category of R-modules for some ring R. We will tend to avoid this formalism, especially because of the following theorem, called the Freyd Embedding Theorem:

Theorem: *Let $\mathcal{C}$ be a small abelian category. Then there exists a ring R with* 1 *(R is not necessarily commutative) and a full, faithful and exact functor $F : \mathcal{C} \to R - mod$, where $R - mod$ denotes the category of left R-modules. In particular, $\mathcal{C}$ is equivalent to a full subcategory of $R - mod$, and this equivalence preserves exact sequences.*

Thus, we may as well work in the category of R-modules for some ring R. This has many benefits, aside from the obvious one that the category of R-modules is rather explicit, while an arbitrary abelian category (at least by definition) is rather abstract. The true benefit of workin in R-modules as opposed to an arbitrary abelian category is that in R-mod we can "chase diagrams" since objects of R-mod are, in particular, sets. Thus, throughout this note, we will work in the category of R-modules for some unital ring R.

1. Complexes and (co)homology

A **homological complex** of R-modules is a sequence $(M_\bullet, d)$ of R-modules, indexed by $n \in \mathbb{Z}$, together with morphisms

$$d : M_{n+1} \to M_n,$$

such that the composition

$$M_{n+1} \xrightarrow{d} M_n \xrightarrow{d} M_{n-1}$$

is trivial. We can view a complex as a graded R-module $M_\bullet = \bigoplus_n M_n$, and d as a morphism $M_\bullet \to M_\bullet$ which is homogeneous of degree 1, meaning that $d(M_{n+1}) \subset M_n$, and such that $d^2 = 0$.

If $M_* = (M_*, d)$ is a complex, then the n-th homology group of M_* is defined to be:

$$\mathrm{H}_n(M_*) = \ker(M_n \to M_{n-1})/\operatorname{im}(M_{n+1} \to M_n).$$

A **cohomological complex** of R-modules is a sequence $(M^\bullet, d)$ of R-modules, indexed by $n \in \mathbb{Z}$, together with morphisms

$$d : M^n \to M^{n+1}$$

such that $d^2 = 0$. The n-th cohomology group of such a complex is defined to be

$$\mathrm{H}^n(M^*) = \ker(M^n \to M^{n+1})/\operatorname{im}(M^{n-1} \to M^n).$$

The map d is usually called a **differential**. We will usually consider homology resp. cohomology as a complex with zero differential, and write it as $H_*(M_*)$ resp. $H^*(M^*)$.

The idea of homology is to detect how far a complex, say a cohomological complex:

$$\cdots \to M^{n-1} \to M^n \to M^{n+1} \to \cdots$$

is from being exact. Indeed, one has $H^n(M^*) = 0$ for all n if and only if this complex is an exact sequence.

There is no essential difference between homological and cohomological complexes (in fact, a homological complex in R-mod is the same as a cohomological complex in $(R - mod)^{\mathrm{op}}$). So we may as well just work with one kind of complex. And because we will eventually be interested in group cohomology, we will mostly work with cohomological complexes.

A **morphism** of cohomological complexes $f : M^* \to N^*$ is just a collection of R-mod moprhisms $f^n : M^n \to N^n$ such that $d \circ f^n = f^{n+1} \circ d$. It is easy to see that a morphism of complexes $f : M^* \to N^*$ inudces a morphism on cohomology groups $f^* : H^*(M^*) \to H^*(N^*)$. This lets us define the *category* of (cohomological) complexes in the obvious way, denoted $C^*(R - mod)$ or just $C^*(R)$. We make a few observations:

(1) First, one has an embedding $R - mod$ into $C^*(R)$, defined by $M \mapsto M$, where M is considered as a complex concentrated in degree 0, with zero differential.
(2) For each n, H^n is a functor $C^*(R) \to R - mod$.
(3) H^* is a functor $C^*(R) \to R - mod$ as well, but we can also consider H^* as a functor $C^*(R) \to C^*(R)$, since H^* is considered as a cohomological complex with trivial differential.
(4) More generally, the category of graded R-modules embeds as a full subcategory in the category of complexes (the full subcategory of complexes with trivial differential).

The following notation is more-or-less standard:

(1) The submodule of cocycles $Z^n(M^*) = \ker(M^n \to M^{n+1})$.
(2) The submodule of coboundaries $B^n(M^*) = \operatorname{im}(M^{n-1} \to M^n)$.
(3) One defines cycles Z_n and boundaries B_n for homological complexes in a similar way.

2. Examples of Complexes

2.1. Let S be a set. For $i = 0, 1, 2, \ldots$ let E_i be the free $\mathbb{Z}$-module generated by $(i+1)$-tuples $(x_0, \ldots, x_i)$ with $x_j \in S$. Thus, such $(i+1)$-tuples form a basis of E_i over $\mathbb{Z}$:

$$E_i = \bigoplus_{(x_0,\ldots,x_i)\in S^{i+1}} \mathbb{Z} \cdot (x_0, \ldots, x_i).$$

We have differential:

$$d_{i+1} : E_{i+1} \to E_i$$

defined by

$$d_{i+1}(x_0, \dots, x_i) = \sum_{j=0}^{i+1} (-1)^j (x_0, \dots, \hat{x}_j, \dots, x_i)$$

extended linearly. This forms a homomological complex.

Now suppose that M is an arbitrary abelian group. And consider:

$$C^i = \mathrm{Hom}(E_i, M)$$

with the differential given by the fact that $\mathrm{Hom}(\bullet, M)$ is a functor. Then C^i is a cohomological complex.

This example will reappear when we talk about projective/free resolutions of modules.

2.2. Let A be a commutative ring, and let $x_1, \dots, x_n \in A$ be given. Let's consider: $K_i(x) := \wedge^i(Ae_1 \oplus \dots \oplus Ae_n)$. Define

$$d : K_p(X) \to K_{p-1}(X)$$

by the formula:

$$d(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} \cdot x_{i_j} \cdot e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_p}.$$

This is a homological complex, called the *Koszul complex* associated to (X). In most situations, it essentially depends only on the ideal generatex by (X), but I won't give the details here.

For instance, let's think about the case where we have only one element $x \in A$. Then this complex becomes:

$$0 \to A \xrightarrow{x} A \to 0.$$

2.3. Let M be a smooth (real) manifold, and let Ω^i denote the module of smooth differential i-forms on M. I.e. Ω^i is generated by differential forms on M, which locally look like

$$f \cdot dx_I$$

for some I of size i, and some smooth function f. So for example Ω^0 is the smooth functions on M, Ω^1 the smooth 1-forms, etc.

We define a differential $\Omega^i \to \Omega^{i+1}$, called the *exterior derivative*, which is defined, locally, as follows. Locally an element $\omega \in \Omega^r$ looks like

$$\omega := \sum_{|I|=r} f_I \cdot dx_I$$

the differential on this local representation is defined by

$$d\omega = \sum_{|I|=r} \sum_{j=1}^{r} \frac{\partial f_I}{\partial x_j} \cdot dx_j \wedge dx_I.$$

Then (Ω^*, d) forms a cohomological complex. The cohomology of this complex is called the **de-Rham** cohomology of M (with coefficients in $\mathbb{R}$).

3. Exact Sequences

A **short exact sequence** of complexes is a sequence of morphisms of complexes:

$$0 \to M_1^* \to M_2^* \to M_3^* \to 0$$

such that for all n, the restricted sequence

$$0 \to M_1^n \to M_2^n \to M_3^n \to 0$$

is exact.

One of the most important uses for homology, is that a short exact sequence of complexes yields a *long exact sequence* of homology groups. To be precise, we define a map

$$\delta : \mathrm{H}^n(M_3^*) \to \mathrm{H}^{n+1}(M_1^*)$$

as follows.

Recall the snake Lemma:

Lemma 3.1. *Suppose we have a commutative diagram of R-modules with exact rows:*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

Then one has a long exact sequence:

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{d} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c).$$

Now the map $\delta : \mathrm{H}^n(M_3^*) \to \mathrm{H}^{n+1}(M_1^*)$ is the map d in the snake lemma associated to the diagram:

$$\begin{tikzcd} 0 \arrow[r] & M_1^n \arrow[r] \arrow[d, "d"'] & M_2^n \arrow[r] \arrow[d, "d"'] & M_3^n \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & M_1^{n+1} \arrow[r] & M_2^{n+1} \arrow[r] & M_3^{n+1} \arrow[r] & 0 \end{tikzcd}$$

So that we obtain a map

$$d : \ker(M_3^n \to M_3^{n+1}) \to \operatorname{coker}(M_1^n \to M_1^{n+1}).$$

On the one hand, we can check that an element of $\operatorname{im}(M_3^{n-1} \to M_3^n)$ is killed under this map, so that this map descends to a map

$$d : \mathrm{H}^n(M_3^*) \to \operatorname{coker}(M_1^n \to M_1^{n+1}).$$

And on the other hand, note that $\mathrm{H}^{n+1}(M_1^*) \subset \operatorname{coker}(M_1^n \to M_1^{n+1})$, and that the image of d is contained in this submodule. In total, we obtain

$$\delta : \mathrm{H}^n(M_3^*) \to \mathrm{H}^{n+1}(M_1^*).$$

This is defined as follows: Let $a \in Z^n(M_3^*)$ be a representative of an element of $\mathrm{H}^n(M_3^*)$. Then $\delta(a) = d(\tilde{a})$ where $\tilde{a} \in M_2^n$ is a lift of a under the surjective map $M_2^n \to M_3^n$.

Patching everything together, one has the following proposition:

Proposition 3.2. *Given a short exat sequence of complexes* $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$, *one has a long exact sequence of cohomology modules:*

$$\cdots \to \mathrm{H}^n(M_1) \xrightarrow{f} \mathrm{H}^n(M_2) \xrightarrow{g} \mathrm{H}^n(M_3) \xrightarrow{d} \mathrm{H}^{n+1}(M_1) \to \cdots$$

Proof. This essentially just follows from the snake lemma, exercise. □

4. Exact Functors and Additive Functors

We say that a functor F from the category of R-modules to abelian groups is an **additive functor** if the induced map on Hom-sets is a homomorphism of abelian groups. For example, $\mathrm{Hom}(\bullet, A)$ and $\hom(A, \bullet)$ are both additive functors. Similarly, $\bullet \otimes_A M$ and $M \otimes_A \bullet$ are additive functors.

The main important property of additive functors is that they are compatible with complexes. I.e. if (M^*) is a (co)-chain complex, and F is an additive covariant functor, then $(F(M^*))$ is a co-chain comlpex. If F is a contravariant additive functor, then $F(M^*)$ is a chain complex, etc.

Note that an additive functor need not preserve exact sequences. I.e. if M^* is exact, then $F(M^*)$ need not be exact. This leads us to the next definition.

Let F be an additive functor from the category of R-modules to the category of abelian groups. We say that F is *exact* provided that any exact sequence of R-modules:

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

induces an exact sequence of abelian groups:

$$0 \to F(M_1) \to F(M_2) \to F(M_3) \to 0.$$

If F is contravariant, a similar definition holds. Namely, F is exact if the induced complex

$$0 \to F(M_3) \to F(M_2) \to F(M_1) \to 0$$

is exact.

We say that a (covariant) F is left-exact if the induced sequence

$$0 \to F(M_1) \to F(M_2) \to F(M_3)$$

is exact. We say that a contravariant F is left exact if

$$0 \to F(M_3) \to F(M_2) \to F(M_1)$$

is exact. And one can similarly define right-exact functors and contravariant functors (left to the reader).

Unfortunately (or furtunately, depending on your point of view), the most natural functors are usually not exact. For instance, if M is an R-module, then $\mathrm{Hom}(\bullet, M)$ and $\mathrm{Hom}(M, \bullet)$ are only left-exact. Similarly, the functor $\bullet \otimes_R M$ is right exact.

This leads us to the following definitions:

(1) We say that M is injective provided that $\mathrm{Hom}(\bullet, M)$ is exact.
(2) We say that M is projective provided that $\mathrm{Hom}(M, \bullet)$ is exact.
(3) We say that M is flat, provided that $\bullet \otimes_R M$ is exact. (we will come back to this one later).

As we can see, working with projective/injective modules is highly desirable. So we would like to "approximate" an arbitrary R-module by injective/projective modules.

First we prove some basic properties concerning projective modules.

Lemma 4.1. *Free modules are projective.*

Theorem 4.2. *The following are equivalent:*

(1) P *is projective.*
(2) $\mathrm{Hom}(\bullet, P)$ *is exact.*
(3) Every surjective map $B \twoheadrightarrow P$ *has a splitting.*
(4) P *is a direct summand in every module of which it is a quitient.*
(5) P *is a direct summand of a free module.*

Proof. (1) $\Leftrightarrow$ (2) there is nothing to prove. Assume (2) $\Rightarrow$ (3) follows by considering the short exact sequence

$$0 \to K \to B \to P \to 0.$$

This also implies (4), and (5) follows by letting B be a free presentation of P.

The non-trivial part is (5) implies (1). In fact, we prove the following stronger lemma:

Lemma 4.3. *Suppose that* $P \cong \bigoplus P_i$. *Then* P *is projective iff* P_i *are all projective.*

Proof. If P_i are projective then this is the easy case. Conversely, for simplicity, assume that $P \cong A \oplus B$.

Let $A \to C$ be a morphism and $B \to C$ be surjective. Consider the map $A \oplus B \to C$ where the map on B is the zero map. By assumption, this has a lifting to B, so we get a lifting of $A \to C$. □

□

5. Inective/Projective Resolutions

Lemma 5.1. *A free* R*-module is projective.*

Proof. Easy, from the definition, or the universal property of free modules. □

As a consequence, we obtain the following

Lemma 5.2. *Let* M *be any* R*-module. Then there exists a surjective morphism* $P \to M \to 0$ *of* R*-modules, from a projective* R*-module* P *onto* M.

Proof. We can consider the "standard" free resolution:

$$R^(M) \to M \to 0$$

sending, for $m \in M$, the element $[m] \in R^(M)$ to $m \in M$. This is clearly R-linear and surjective. □

If M is an R-module, we say that a **projective-resolution of** M is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

Using the lemma above it is fairly easy to produce a projective (in fact a free) resolution of any R-module M, as follows:

(1) First, consider the free R-module generated by M, together with the canonical surjective map:

$$R^{(M)} \to M \to 0.$$

(2) Next, consider the kernel of the map above, say K, so that we have a short exact sequence

$$0 \to K_1 \to R^{(M)} \to M \to 0.$$

(3) Next, consider the surjective map

$$R^{(K_1)} \to K_1$$

defined in a similar way to the way we defined $R^{(M)} \to M$. Now we can splice the two sequence together to form

$$R^{(K_1)} \to R^{(M)} \to M \to 0$$

in the obvious way.

(4) Now consider the kernel K_2 of $R^{(K_1)} \to R^{(M)}$, etc...

To summarize, we produced a *free* resolution of M:

$$\cdots \to R^{(K_2)} \to R^{(K_1)} \to R^{(M)} \to M \to 0.$$

Note that if M is already free, e.g. if $M = R$, then we can form a very simple free resolution:

$$0 \to M \to M \to 0.$$

So, if we think about projective/free resolutions as a *replacement* of M, then if M is already free/projective, it doesn't need to be replaced.

5.1. Consider, for example, the ring $A = k[x_1, \ldots, x_n]$. Then the Koszul complex $K_*(x_1, \ldots, x_n)$ is a *free resolution* of k, as an A-module via the surjective map

$$k[x_1, \ldots, x_n] \twoheadrightarrow k$$

defined by $x_i \mapsto 0$.

6. Injective Resolutions

What about injective reslutions? Let M be an R-module. An **injective resolution of** M is an exact sequence of the form

$$0 \to M \to I_1 \to I_2 \to \cdots$$

where each I_i is injective. To form injective resolutions, we need to be able to realize any R-module as a submodule of an injective module (this is dual to the picture that any module has a projective presentation as a surjective map $P \to M \to 0$ from some projective module P).

WE first work with the category of abelian groups, i.e. $\mathbb{Z}$-modules, and show that this property holds true in that category. Let M be an abelian group. We denote by $M^\vee$ the *dual group*:

$$M^\vee := \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}).$$

And note that we have a canonical homomorphism:

$$M \to M^{\vee\vee}.$$

It is easy to see that this map is injective. Now take a free presentation of M:

$$F \to M \to 0$$

by a free abelian group $F = \mathbb{Z}^{(I)}$ for some I. Then the dual of $F \to M \to 0$ is

$$0 \to M^\vee \to F^\vee$$

(recalling that $\mathrm{Hom}(\bullet, \mathbb{Q}/\mathbb{Z})$ is left-exact) and thus we obtain an injective map

$$0 \to M \to F^\vee.$$

We claim that $F^\vee$ is injective.

Lemma 6.1. *If T is a divisible abelian group, then T is injective in the category of abelian groups.*

Proof. Let $M' \subset M$ be a subgroup and let $f : M' \to T$ be a homomorphism. Let $x \in M$ be given. We need to extend f to (M', x), the module generated by M' and x in M. If x is free over M' (i.e. $(M', x) = M' \oplus (x)$) then we can define $f(x)$ to be any element of T.

Suppose, on the other hand, x is torsion over M', i.e. $m \cdot x \in M'$ for some $m \in \mathbb{Z}$. Let d the order of $x + M'$ in M/M'. By hypothesis, there exists an element $u \in T$ such that $d \cdot u = f(d \cdot x)$. We define

$$f(z + nx) = f(z) + nu$$

for $n \in \mathbb{Z}$, $z \in M'$. It is straightforward to check that this is well-defined (i.e. independent of the choice of x as a generator of (M', x) over M').

To conclude the proof of the Lemma, we apply Zorn's Lemma. I.e. we consider the collection of submodules N of M containing M', together with lifts g of f to N. Call such an element (N, g). We order these by saying $(N, g) \leq (N', g')$ iff $N \subset N'$ and $g'|_N = g$. It is easy to see that any chain in this collection has an upper bound in this collection. So let's take a maximal element (N, g) in this collection, and suppose that $N \neq M$. Then take $x \in M \smallsetminus N$, and repeat the argument above with (N, x) to get a contradiction. $\square$

Now note that $F^\vee$ is divisible, in the notation above, since it is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{(I)}$ for some I. So we proved that any abelian group is a subgroup of a divisible (hence injective) abelian group.

Now we need to extend this to arbitrary R-modules for an arbitrary ring R. For a ring R and a group T, we define a module structure on $\mathrm{Hom}_{\mathbb{Z}}(A, T)$ by saying that

$$(af)(b) = f(ab).$$

For any A-module M, we have a (functorial) isomorphism of abelian groups:

$$\mathrm{Hom}_{\mathbb{Z}}(M, T) \xrightarrow{\cong} \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, T))$$

defined by

$$\phi \mapsto (m \mapsto (a \mapsto \phi(am))).$$

The inverse is given by

$$f \mapsto (m \mapsto f(m)(1)).$$

Lemma 6.2. *If T is a divisible abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A, T) =: I$ is an injective A-module.*

Proof. Supose that $0 \to M \to N$ is an injective morphism of A-modules, and consider the induced map

$$\mathrm{Hom}_A(N, I) \to \mathrm{Hom}_A(M, I).$$

By the remark above, we see that this is equivalent to a mrphis,

$$\mathrm{Hom}_{\mathbb{Z}}(N, T) \to \mathrm{Hom}_{\mathbb{Z}}(M, T)$$

which is surjective by what we proved for abelian groups. □

Now to complete the fact that any A-module can be embedded in an injective A-module, we note that we can embed M into a divisible abelian group, say T, as we did for plain abelian groups. Say $f : M \hookrightarrow T$ is such an embedding. Now consider the map

$$M \to \mathrm{Hom}_{\mathbb{Z}}(A, T)$$

defined by

$$m \mapsto (a \mapsto f(am))$$

this is injective, and $\mathrm{Hom}_{\mathbb{Z}}(A, T)$ is injective by the lemma above. This completes the proof that any A-module can be embedded into an injective one, and thus that any A-module has an injective resolution in the category of A-modules.

7. Homotopy of Complexes

Suppose that M^* and N^* are two (cohomological/cochain) complexes. And suppose that we are given $f, g : M^* \to N^*$ two morphisms. A **homotopy** $h : f \to g$ is a collection of maps

$$h^n : M^n \to N^{n-1}$$

such that $f^n - g^n = d \circ h^n + h^{n+1} \circ d$.

Fact 7.1. *Consider the maps $H(f)$ and $H(g)$ inducd by f, g on cohomology. If there exists a homotopy $h : f \to g$, then one has $H(f) = H(g)$.*

Proof. Diagram chase. □

Exercise 7.2.
(1) Homotopy of morphisms is an equivalence relation.
(2) If F is an additive functor (i.e. $F(0) = 0$), and f, g are homotopic, then $F(f)$ and $F(g)$ are homotopic.
(3) We can also define a homotopy of chain (homological) complexes as the dual of a homotopy of co-chain complexes.
(4) If F is an additive functor, and f is homotopic to g, then $F(f)$ is homotopic to $F(g)$.

We say that two complexes are **homotopic** if they are isomorphic in the category where morphisms are homotopy-equivalence classes of chain maps. I.e. if $C^*(R)$ denotes the category of cochain complexes of R-modules, then the **homotopy category** associated to $C^*(R)$ is the category whose objects are cochain complexes, and morphisms are

$$\mathrm{Hom}^{\mathrm{homotopy}}(C^*, D^*) = \mathrm{Hom}^{\mathrm{cochain}}(C^*, D^*)/\mathrm{homotopy}.$$

In other words, C^* and D^* are homotopic complexes if there exist morphisms

$$f : C^* \to D^*, \quad g : D^* \to C^*$$

such that $f \circ g \sim \mathbf{1}$ and $g \circ f \sim \mathbf{1}$ (where $\sim$ denotes homotopy equivalence). Note that if C^* and D^* are homotopic, say f is a homotopy equivalence between them, then f induces an isomorphism

$$H^*(f) : H^*(C^*) \to H^*(D^*).$$

8. Uniqueness of Resolutions

Suppose now that M is an R-module. Recall that a projective resolution is an exact sequence

$$\cdots \to P_1 \to P_0 \to M \to 0.$$

We will now consider the *complex*

$$\cdots \to P_1 \to P_0 \to 0.$$

and call (P_*) a projective resolution of M by abuse of notation. I.e. a projective resolution $(P_*)_{*\geq 0}$ is a chain complex with a given isomorphism

$$H_0(P_*) \xrightarrow{\cong} M$$

such that each term P_i is projective.

Lemma 8.1. *Suppose that $(P_*)_{*\geq 0}$ is a chain complex of projective modules, and that $(A_*)_{*\geq 0}$ is exact. Let $\phi : H_0(P_*) \to H_0(A_*)$ be a morphism. Then there exists a chain morphism $f : (P_*) \to (A_*)$ inducing ϕ. Moreover, if g is another chain morphism inducing ϕ, then f and g are homotopic. Dually, can do this with cochain complex of injective modules, and a morphism $\phi : (A^*) \to (I^*)$, etc.*

Proof. diagram. $\square$

Corollary 8.2. *If (P_*) and (P'_*) are two projective resolutions of the same module M, then (P_*) and (P'_*) are homotopy equivalent. Dually, if (I^*) resp $((I')^*)$ are two injective resolutions of the same module, then (I^*) and $((I')^*)$ are homotopy equivalent.*