DERIVED FUNCTORS
MATH 250B

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1. Derived Functors

We are now prepared to introduce the left/right **derived** functors associated to a (covariant) additive functor F from the category of R-modules to the category of abelian groups. One can easily carry out the same definition for a contravariant functor, but we leave the details to the reader.

As such, we fix such a covariant additive functor F.

1.1. Left Derived Functors.

Here we define the left derived functors of F, denoted by $L_i F$. The construction proceeds as follows.

(1) For a module M, let $(P_*)_*$ be a projective resolution of M.
(2) Consider the complex $(F(P_*))_*$.
(3) Define $L_n F(M) := \mathrm{H}_n(F(P_*))$.

To check that this is well defined, we first check that the definition of $L_n F(M)$ doesn't depend on the choice of projective resolution P. We denote the modules constructed by P' by $L'_n F$. As such, suppose that P' is another projective resolution. By a theorem proved before, we know that there is a chain map $P \to P'$ inducing the identity on M. Moreover, this chain map is unique up-to homotopy. Similarly, we have a chain map in the other direction $P' \to P$ inducing the identity on M. Recall that if f, g are homotopic maps, and F is additive, then $F(f)$ and $F(g)$ are also homotopic.

From this, we deduce that there are maps

$$L_* F(M) \to L'_* F(M), L'_* F(M) \to L_* F(M)$$

such that the composition of two in any direction is the identity. I.e. we have isomorphisms between $L_* F(M)$ and $L'_* F(M)$, but it is important to note that this isomorphism is induced by a homotopy equivalence on the associated projective resolutions.

Now to prove that $L_* F$ is a functor. Suppose that $M \to M'$ is a morphism, and let P resp. P' be projective resolutions of M resp. M'. by the same theorem, there exists a chain map $P \to P'$ inducing the morphism $M \to M'$. From this it follows that we obtain an induced morphism

$$L_* F(M) \to L_* F(M').$$

Moreover, if we change the projective resolutions P resp. P', then the morphism $L_* F(M) \to L_* F(M')$ is compatible with the isomorphisms we obtain by changing the resolutions (why?).

To be completely preicse, what we're doing here is this. For each module, we choose once and for all a projective resolution, and compute the left-derived funcots using this projective resolution. This yields a construction of the left-derived functos $L_* F$. If we change these choices of projective resolutions, we get another left-derived functor L'_*F, but this other left-derived functor is **naturally isomorphic** to L_*F.

2. Right Derived Functors

Here we define the right derived functors of F, denoted R^iF. the construction proceeds as follows:

(1) For a module M, let I be an injective resolution of M.
(2) Consider the complex $F(I)$.
(3) Define $R^iF(M) = H^i(F(I))$.

Using the "injectives" version of the main theore concerning chain maps discussed before, we see that indeed R^iF is a functor, and that R^iF doesn't depend on the choice of injective resolution (up-to a functorial isomorphism).

3. Contravariant Functors

The left/right derived functors of a contravariant functor are defined in a similar way. The only distinction is that the right-derived functors are computed using a **projective** resolution, while the left derived functors are computed using an **injective** resolution, which is the opposite to the case of a covariant functor. The details are left to the reader.

4. Left/Right Exact Functors

We will be primarily interested in considering the left/right derived functors in the situation where F is right/left exact.

4.1. Left Derived Functors.

Suppose that F is left-exact, additive and covariant.

Lemma 4.1. *One has an isomorphism of functors $F \cong R^0F$.*

Proof. Let I be a projective resolution of M. I.e. we have an exact sequence

$$0 \to M \to I^1 \to I^2 \to \cdots$$

And we note that $F(M) = \ker(F(I^0) \to F(I^1))$ since F is left-exact. But on the other hand, we note that $R^0F(M) = \mathrm{H}^0(F(I)) = \ker(F(I^0) \to F(I^1))$. $\square$

Lemma 4.2. *Suppose that I is an injective module. Then $R^iF(I) = 0$ for all $i \geq 1$.*

Proof. Follows from the definition, since $0 \to I \to I \to 0$ is an injective resolution of I. $\square$

Lemma 4.3. *The functors R^iF are additive.*

Proof. A functor G is additive if and only if $G(A \oplus B) = G(A) \oplus G(B)$ (functorially in A and B). Now use the fact that if I resp. J is an injective resolution of A resp. B, then $I \oplus J$ is an injective resolution of $A \oplus B$. Details are an exercise. $\square$

4.2. Right Derived Functors.

Now suppose that F is right exact, additive and covariant. We obtain the analogous lemmas to the previous case.

Lemma 4.4. *One has an isomorphism of functors $F \cong L_0F$.*

Lemma 4.5. *If P is projective then $L_iF(P) = 0$ for all $i \geq 1$.*

Lemma 4.6. *The functors L_iF are additive.*

4.3. **Contravariant Functors.** Similar statements hold for contravariant functors.

5. Long Exact Sequences

The main reason we care about derived functors is because they yield long exact sequences associated to left/right exact functors.

6. Left exact functors

Suppose that F is left exact and covariant, and additive.

Theorem 6.1. *Let $0 \to A \to A' \to A'' \to 0$ be a short exact sequence of modules. Then there are connecting morphisms*

$$R^i F(A'') \to R^{i+1} F(A)$$

which fit into a long exact sequence

$$0 \to F(A) \to F(A') \to F(A'') \to R^1 F(A) \to \cdots$$

If we have a diagram of short exact sequences (...) then we get an induced diagram of long exact sequences (...)

Proof. The idea is to choose projective resolutions P, P', P'' of A, A', A'' such that there is an exact sequence of complexes:

$$0 \to P \to P' \to P'' \to 0.$$

Since P'' is projective, this sequence is split, and therefore $P' = P \oplus P''$. Since F is additive, we obtain a short exact sequence of complexes

$$0 \to F(P) \to F(P') \to F(P'') \to 0.$$

Now the theorem would follow from the definition of the right-derived functors using projective resolutions, along with the long exact sequence of cohomology associated to a short exact sequence of complexe.

To find P, P', P'': Choose projective presentations P_0 of A and P_0'' of A''. Define $P_0' = P_0 \oplus P_0''$ and note that this is projective. Since P_0'' is projective, there exists a lift of $P_0'' \to A'$ of $P_0'' \to A''$. Thus we obtain a canonical map $P_0' \to A'$ which is the map $P_0 \to A \to A'$ on P_0 and this lift of P_0''. It is easy to check that this is surjective.

Next consider the kernels of $P_i^t \to A^t$ for $t = \varnothing, ', ''$ and proceed to construct projective resolutions in the same way as we construct projective resolutions of a module. $\square$

7. Right Exact Functors

Suppose that F is right exact and covariant, and additive.

Theorem 7.1. *Let $0 \to A \to A' \to A'' \to 0$ be a short exact sequence of modules. Then there are connecting morphisms*

$$F_{i+1} F(A'') \to L_i F(A)$$

which fit into a long exact sequence

$$\cdots \to L_1 F(A'') \to F(A) \to F(A') \to F(A'') \to 0.$$

If we have a diagram of short exact sequences (...) then we get an induced diagram of long exact sequences (...)

Proof. similar. □

7.1. **Contravariant functors.** We can formulate and prove similar statements concerning contravariant left/right exact functors.

8. Natural Transformations

Suppose that $F \to G$ is a natural transformation of additive covariant functors. Then one obtains natural transformations $L_*F \to L_*G$ and $R^*F \to R^*G$ (details left to the reader). If F is left/right exact, then these natural transformations yield a giant commutative diagram of long exact sequences associated to F resp. G, as in the two theorems above (drawn in class).

9. Long Exact Sequences

Theorem 9.1. *Suppose that $F \to F' \to F''$ is a sequence of functors which is exact on projectives. Then for every module A, there is a connecting map*

$$L_{i+1}F''(A) \to L_iF(A))$$

which fits into a long exact sequence

$$\cdots \to L_1F''(A) \to L_0F(A) \to L_0F'(A) \to L_0F''(A) \to 0.$$

Proof. Let P be a projective resolution. Then the sequence

$$0 \to F(P) \to F(P') \to F(P'') \to 0$$

is exact, etc. □

Theorem 9.2. *Suppose that $F \to F' \to F''$ is a sequence of functors which is exact on injectives. Then for every module A, there is a connecting map*

$$R^iF''(A) \to R^{i+1}F(A)$$

which fits into a long exact sequence

$$0 \to R^0F(A) \to \cdots$$

9.1. **Contravariant.** There are analogous statements about contravariant functors.

10. Ext Functors

We define $\mathrm{Ext}^n_P(\bullet, N)$ to be $(R^n \operatorname{Hom}(\bullet, N))$. We define $\mathrm{Ext}^n_I(M, \bullet)$ to be $R^n \operatorname{Hom}(M, \bullet)$. I.e. Ext_P is computed using a projective resolution of $\bullet$ while Ext_I is computed using an injective resolution of $\bullet$.

Our goal for this section will be to prove the equivalence of $\mathrm{Ext}_P(M, N)$ and $\mathrm{Ext}_I(M, N)$, functorially both in M, N. This will be a sketch, the details are left to the reader (and can be found in most homological algebra texts).

10.1. **Using projectives.** We need to use some facts concerning Ext_P:

Proposition 10.1. *Ext_P is a bifunctor, meaning that $\mathrm{Ext}_P(M, N)$ is functorial (contravariantly) in M and covariantly in N.*

Proof. We know that it is functorial in M from what we already proved. To see that it is functorial in N, note that a morphism $N_1 \to N_2$ induces a morphism of functors

$$\mathrm{Hom}(\bullet, N_1) \to \mathrm{Hom}(\bullet, N_2).$$

Now one can prove that a morphism of functors induces a morphism of derived functors (see homework 3). Thus we obtain a morphism of functors

$$\mathrm{Ext}^n_P(\bullet, N_1) \to \mathrm{Ext}^n_P(\bullet, N_2)$$

and if we apply M to this natural transformation, this shows that $\mathrm{Ext}_P(M, \bullet)$ is a functor. $\square$

Lemma 10.2. *If P is projective and I is injective then*

$$\mathrm{Ext}^n_P(P, B) = 0 = \mathrm{Ext}^n_P(A, I)$$

for $n \geq 1$.

Proof. Concerning P, this is clear using the projective resolution $0 \to P \to 0$ of P. Converning I, this is clear since the functor $\mathrm{Hom}(\bullet, I)$ is exact. $\square$

Proposition 10.3. *Let $N_1 \to N_2 \to N_3$ be a short exact sequence, then the sequence of functors*

$$\mathrm{Hom}(\bullet, N_1) \to \mathrm{Hom}(\bullet, N_2) \to \mathrm{Hom}(\bullet, N_3)$$

is short exact on projectives.

In particular, for any module A, one obtains a long exact sequence

$$0 \to \mathrm{Hom}(A, N_1) \to \cdots$$

Proof. trivial. $\square$

One has analogous statements concerning Ext_I, left to the reader. I.e.

(1) Ext_I is a bifunctor, contravariant in the first coordinate and covariant in the second.
(2) $\mathrm{Ext}^n_I(P, B) = 0 = \mathrm{Ext}_I(A, I)^n$ for P projective and I injective, and $n \geq 1$.
(3) A short exact sequence of M's induces a long exact sequnence of Ext_I's

11. Equivalence of two notions of ext

The main goal of this section is to prove that the two bifunctors

$$\mathrm{Ext}^*_P(\bullet, \bullet), \quad \mathrm{Ext}^*_I(\bullet, \bullet)$$

are naturally equivalent (isomorphic functorially). We only give a sketch.

We will construct a map $\mathrm{Ext}^*_P(\bullet, \bullet) \to \mathrm{Ext}^*_I(\bullet, \bullet)$, inductively for every $*$, which will become our isomorphism. The base case $* = 0$ is trivial, it is just the identity. The proof for $* > 0$ follows by a sort-of *devissage* argument, i.e. by proving that $\mathrm{Ext}^{*+1}(A, B)$ is isomorphic to $\mathrm{Ext}^*(C, D)$ for certain ("canonical up-to homotopy") modules C, D.

Let A, B be given. We will show how to construct the isomorphism $\mathrm{Ext}^*_P(A, B) \to \mathrm{Ext}^*_I(A, B)$. Choose an injective presentation

$$0 \to B \to I \to C \to 0$$

and consider the long exact sequence induced by this:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A,I) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(A,C) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_P(A,B) \arrow[r] \arrow[d, dotted] & \mathrm{Ext}^1_P(A,I)=0 \\
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(A,I) \arrow[r] & \mathrm{Hom}(A,C) \arrow[r] & \mathrm{Ext}^1_I(A,B) \arrow[r] & \mathrm{Ext}^1_I(A,I)=0
\end{tikzcd}$$

We define $\mathrm{Ext}^1_P(A,B) \to \mathrm{Ext}^1_I(A,B)$ to be the unique map which makes the diagram above commute, and note that this map is then an isomorphism.

Having defined $\mathrm{Ext}^i_P(A,B) \cong \mathrm{Ext}^i_I(A,B)$ inductively for all $i \neq n$, we now define it for $n+1$. Note that the maps

(1) $\mathrm{Ext}^n_P(A,C) \to \mathrm{Ext}^{n+1}_I(A,B)$
(2) $\mathrm{Ext}^n_I(A,C) \to \mathrm{Ext}^{n+1}_I(A,B)$

are isomorphisms, again by the long exact sequences and the fact that $\mathrm{Ext}^*_*(\bullet, I) = 0$.

We define $\mathrm{Ext}^{n+1}_P(A,B) \to \mathrm{Ext}^{n+1}_I(A,B)$ to be the unique isomoprhism making the following diagram commute.

$$\begin{tikzcd}
\mathrm{Ext}^n_P(A,C) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^{n+1}_P(A,B) \arrow[d, dotted] \\
\mathrm{Ext}^n_I(A,C) \arrow[r] & \mathrm{Ext}^{n+1}_I(A,B)
\end{tikzcd}$$

We leave it to the reader that the isomorphisms $\mathrm{Ext}^*_P(A,B) \cong \mathrm{Ext}^*_I(A,B)$ are functorial in both A and B, and that the associated long exact sequences are also isomorphic, in the obvious sense.

12. Ext vs. Extensions

From the homework, we know that $\mathrm{Ext}^1(M,N)$ classifies equivalence classes of extensions. I.e. an extension of M by N is a short exact sequence

$$0 \to N \to H \to M \to 0$$

of modules. The set of equivalence classes of such extensions is denoted by $E(M,N)$. In this section, we describe the functorial properties of $E(M,N)$, which exhibit the fact that $\mathrm{Ext}^1(M,N)$ is a bifunctor.

First, suppose that $N \to N'$ is a morphism and that $[H]$ is an extension. Then the pushout $[H \coprod_N N']$ is an extension of M by N'. This yields a map $E(M,N) \to E(M,N')$.

Next suppose that $M' \to M$ is a morphism, then the pullback $[M' \times_M H]$ is an extension of M' by N. We therefore get a map $E(M,N) \to E(M',N)$. One can check that these maps correspond to the maps $\mathrm{Ext}^1(M,N) \to \mathrm{Ext}^1(M,N')$ resp. $\mathrm{Ext}^1(M,N) \to \mathrm{Ext}^1(M',N)$ given by the bifunctoriality of $\mathrm{Ext}^1(\bullet,\bullet)$.

What about the group structure of $\mathrm{Ext}^1(M,N)$? This is described via a construction called the Baer Sum. Suppose that H and H' are two extensions. We construct the Baer Sum of H and H', defined as

$$\Gamma = H \times_M H'$$

This is an extension of $N \times N$ by M. I.e. this yields a map

$$E(M,N) \times E(M,N) \to E(M, N \times N)$$

To obtain an extension of M by N, we consider the morphism $N \oplus N \to N$ just given by adding. I.e. $n \oplus n' \mapsto n + n' \in N$. Thus we obtain a morphism by functoriality

$$E(M, N \oplus N) \to E(M, N).$$

The Baer sum is defined to be the composition

$$E(M, N) \times E(M, N) \to E(M, N \times N) = E(M, N \oplus N) \to E(M, N).$$

It is left as an exercise to prove that the Baer sum is well-defined (i.e. equivalent extensions yield equivalent extensions) and that it corresponds to addition in $\mathrm{Ext}^1(M, N)$ under the isomorphism $\mathrm{Ext}^1(M, N) \cong E(M, N)$.