CONCENTRATION PHENOMENA AND APPLICATIONS TO RANDOM MATRICES
TERM PAPER FOR MATH 240

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Abstract. We review applications of the isoperimetric inequality to concentration phenomena. In the case of unitary and orthogonal-valued random matrices, concentration phenomena make it possible to somewhat strengthen the assertions that they are freely independent as their dimensions go to infinity.

The main goal of this paper is to exhibit a powerful technique for dealing with asymptotic behavior of large random matrices. Random matrices are just matrix-valued functions over some measure space; the word random is in their name by analogy with random variables, which are scalar-valued functions over a measure space. Asymptotic behavior of random matrices as their dimensions increase to infinity, is not only interesting in its own right, but also has applications to physics, where random matrices can be used to model random interactions as terms in Hamiltonian operators. For physical applications, it is important to be able to compute the spectrum and spectral distribution as the dimensions of the matrices tend to infinity. *Ad hoc* approaches yielded over the years some results on asymptotic distributions of spectra of certain random matrices (for a review, see e.g. [9].) Recently, Voiculescu in [14] has linked large random matrices with his non-commutative free probability theory. This coupled with techniques of free probability theory allowed simpler computations of "averages" of limit distributions. He also was able to use concentration phenomena, in a manner reviewed in this paper, to strengthen these results, and pass from convergence in "mean" to a stronger form of convergence.

The rest of the paper is divided as follows. The first section is intended as a brief introduction to the free probability theory; free independence is defined, as well as some basic concepts. Theorem 1.1 is a version of the result of Voiculescu ([11]), linking free probability and large random matrices. We then introduce two convolution operations, which are important tools for computing distributions of sums and products of free random variables, in particular, sums and products of random matrices that become asymptotically free. Theorem 1.2 is a result of Speicher ([10]) which uses such methods to compute a limit distribution of a sum of randomly chosen matrices.

The next section is dedicated to the discussion of concentration phenomena, and shows how such phenomena, along the lines discussed in [11] can be applied to give a stronger form of Theorem 1.1, which we state as Theorem 2.1. We also show how to get Theorem 1.2 from Theorem 1.1 and concentration phenomena.

In the next section, we consider two theorems from [7], where the isoperimetric inequality is used to give conditions under which concentration phenomena occur.

Date: February 5, 1996.

We conclude with a computation showing that for the case needed in Theorem 2.1 and its analog involving orthogonal matrices, we indeed have concentration phenomena.

In the appendix, we gather some important facts about the Laplace operator that are used in the rest of the paper.

It is beyond the scope of this paper to give either a comprehensive introduction to concentration phenomena, or the free probability theory (for the latter an excellent introduction is [16].) We also do not touch upon the developing subject of Free Entropy, which is a surprising application of random matrices to free probability theory (see e.g. [15]). However, it is hoped that enough miscellaneous results are assembled together to give a flavor of either subject.

I would like to express my gratitude to Prof. Voiculescu for far-reaching discussions related to these topics as well as some helpful references, and to Prof. Weinstein for giving this assignment and thus stimulating me to research for it, and for some helpful discussions.

1. Free probability theory and large random matrices.

In this section we review some aspects of the free probability theory of Voiculescu ([16]) and in particular the relation to large random matrices. In the section on concentration phenomena we shall consider their application to random matrices, improving the results mentioned in this section.

By a *random variable* one means, in ordinary probability theory, an element of $L^\infty(X, \mu)$ where X is some measure space and μ is a (normalized) measure. Here μ can be viewed as a functional (state) on the von Neumann algebra $L^\infty(X)$, which is normalized, i.e., has value one on the identity element. Classical independence of random variables x and y, considered as functions from Z endowed with a probability measure η, means that Z "splits up" as a direct product $Z = X \times Y$, of two probability measure spaces X and Y with measures μ and ν, so that η is the product measure, and such that x is a random variable on X composed with the projection from $Z = X \times Y \to X$ and y is in a similar way a random variable on Y. In terms of associated L^∞ spaces, this means that $L^\infty(Z)$ splits up as a tensor product $L^\infty(X) \otimes L^\infty(Y)$, such that the state η is the tensor product state $\mu \otimes \nu$ of states on $L^\infty(X)$ and $L^\infty(Y)$, and $x = x' \otimes 1$, $y = 1 \otimes y'$ for some $x' \in L^\infty(X)$ and $y' \in L^\infty(Y)$.

1.1. Free independence.

Commutative von Neumann algebras correspond to algebras $L^\infty(X, \mu)$ for some measure space X with a measure μ, i.e., to (ordinary) probability spaces. In non-commutative probability theory, *non-commutative random variables* are elements of some "non-commutative probability space", i.e. a von Neumann algebra which is not necessarily commutative endowed with a normalized state ϕ (normalized means that $\phi(1) = 1$, 1 being the unit of the algebra.) For our purposes, we do not actually require that the algebra be von Neumann; we treat non-commutative probability spaces as just unital algebras with a certain normalized functional.

Classical independence, as defined for ordinary probability spaces, has a generalization to non-commutative probability spaces: two variables x, y are *classically independent* in a non-commutative probability space (A, ϕ), roughly, if it can be written as a tensor product of two spaces, (B, ψ) and (D, θ), such that $\phi = \psi \otimes \theta$, and $x = x' \otimes 1$, $y = 1 \otimes y'$ for $x' \in B$, $y' \in C$, by analogy with the commutative situation reviewed above. However, independent variables always commute, even in the non-commutative setting.

It turns out that it is possible to define a different kind of independence, called *free independence*, which is in some sense parallel to classical independence, but with the replacement of all tensor products by free products. Such a notion of "free non-commutative probability theory" was introduced in [11].

Definition 1.1. *For a non-commutative probability space (A, ϕ), two subalgebras A_1, A_2 are free, if*

$$\phi(x_1 \dots x_n) = 0 \tag{1}$$

whenever $\phi(x_i) = 0$ and $x_i \in A_{j_i}$ and $j_1 \neq j_2$, $j_2 \neq j_3$, etc. Two elements x, y of a non-commutative probability space (A, ϕ) are called free, if the algebras they generate are free.

Examples of freely independent non-commutative random variables can be obtained by taking free products of noncommutative probability spaces. For free products of algebras we have that if $\mathbb{C}(G)$ denotes the group algebra of a discrete group G, then

$$\mathbb{C}(G * H) = \mathbb{C}(G) * \mathbb{C}(H).$$

A similar statement holds for full and reduced group C^*-algebras, and group von Neumann algebras, once they are suitably defined ([16]). In particular, if $\mathbb{F}(\imath)$ denotes the free group on p generators, then the unitary elements $u_k \in \mathbb{C}(\mathbb{F}(p))$ corresponding to the images of the generators in the left regular representation of the group algebra, are free as non-commutative random variables, if the functional on the algebra is taken to be the usual trace coming from the left regular representation[1]. In fact, the freeness condition expressed in equation (1) can be viewed as a generalization of the situation for free products of groups.

1.2. **Distributions of variables.** Given a non-commutative random variable x, i.e, an element x in a non-commutative probability space (A, ϕ), the distribution of x is defined to be the sequence of numbers

$$\phi(x^k), \quad k \in \mathbb{N};$$

these numbers are called moments of x. For example, when x is self-adjoint, restricting ϕ to the (C^*-)algebra generated by x, we have a certain measure on a subset of the real line[2]. Then the moments as defined here coincide with what is normally called moments of this measure on a subset of the real numbers. Thus to a self-adjoint non-commutative random variable we can associate a measure on the reals; this measure is sometimes also called the distribution of the variable. If $x_1, \dots x_n$ are non-commutative random variables, their joint distribution is defined to be the collection of numbers

$$\phi(x_{i_1}^{k_1} \dots x_{i_s}^{k_s}),$$

[1] If G is a discrete group, then G acts on $\ell^2(G)$, by acting an an orthonormal basis, consisting of sequences δ_h, which are 1 on some element h and zero elsewhere, by $G \ni g : \delta_h \mapsto \delta_{hg^{-1}}$. Then the map $\mathbb{C}(G) \ni T \mapsto \langle T\delta_e, \delta_e \rangle$ where e is the unit of the group, defines a trace on the group algebra, which is what we referred to above.

[2] This subset is, of course, the spectrum of x; ϕ restricts to a trace on $C^*(x)$, and the measure in question is the spectral distribution of x. In fact, this discussion is valid for *normal* elements, i.e., ones that commute with their adjoints. As an example, unitary elements have a naturally associated measure on the unit circle in the complex plane.

$0 < k_r$, $0 < i_r \leq n$, $r = 1, \ldots, s$, $s \in \mathbb{N}$; each such number is called a non-commutative moment. If not all x_i are self-adjoint, one defines the $*$-distribution to be the distribution of the family $(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*)$; in the sequel we shall use the word distribution to mean $*$-distribution. A sequence of n-tuples of random variables

$$\{(x_1^{(i)}, \ldots, x_n^{(i)})\}_i$$

is said to converge in distribution to the n-tuple of random variables $(y_1, \ldots, y_n)$, if each non-commutative moment of $(x_1^{(i)}, \ldots, x_n^{(i)})$ converges to the corresponding moment of $(y_1, \ldots, y_n)$, as $i \to \infty$. We remark that the notion of distribution of variables does not require that we restrict to a particular algebra; for example, it is entirely possible that for different i above, $x^{(i)}$ lie in different and unrelated algebras.

1.3. **Random matrices.** An amazing discovery of Voiculescu is that certain independent large random matrices converge in distribution to a family of free non-commutative random variables. A random matrix over a probability space (X, μ) (μ is a normalized measure: $\mu(X) = 1$) is an element of the non-commutative probability space $M_k \otimes L^\infty(X, \mu)$, endowed with the normalized state $\tau \otimes \mu$ (M_k stands for $k \times k$ complex matrices.) In other words, a random matrix is a matrix-valued L^∞ function on X, and the state is given by

$$\phi(A) = \int \tau(A(x)) d\mu(x).$$

Here τ stands for normalized trace (i.e., it is 1 on the identity element.) A version of Voiculescu's theorem is presented below:

Theorem 1.1. *Let $X_n = U(n)^k$, and $X(n, j) : X_n \to U(n)$ be the unitary random matrix defined by $X(n, j)((u_1, \ldots, u_k)) = u_j$. Let $\{D(n, j)\}_{j=1}^p$ be constant diagonal random matrices over X_n, such that $D(n, j)$ has a limit distribution, and such that* $\sup \|D(n, j)\| < \infty$. *Then the $(k + p)$-tuple*

$$(D(n, 1), \ldots, D(n, p), X(n, 1), \ldots, X(n, k))$$

converges in distribution as $n \to \infty$ to the $(k + p)$-tuple of random variables

$$(d_1, d_2, \ldots, d_p, x_1, \ldots, x_k),$$

in which x_i are free among each other and from the algebra generated by the $d_1, \ldots, d_n$. Moreover, the limit distribution of $X(n, j)$ for each j is that of a Haar unitary, i.e., all moments are zero except for the zeroth (which is 1).

A similar theorem holds for orthogonal rather than unitary matrices.

Asymptotic freeness occurs not only for unitary or othogonal random matrices; matrices whose entries are real or complex independent Gaussian random variables also become asymptotically free (this is in line with certain classical results, such as Wigner's semicircular law; semicircular distributions play the role of Gaussian distribution in free probability.) For details, the reader is referred to [14] and [16]. Such properties of random matrices found applications to free products of von Neumann algebras, where random matrix models can be used to approximate joint distributions of certain elements. For a sample of applications, see [16].

1.4. **Free convolution.** In [11], free probability theory was used to define two convolution operations on measures. If μ, ν are two measures with compact support on the real line, one can pick two free random variables x and y, whose individual distributions are μ and ν[3]. The distributions of xy and $x+y$ depend only on the distributions of x and y. Thus taking the distribution of $x+y$, we obtain a new measure on the reals, of compact support, which is called, the additive convolution of μ and ν. If x, y are of special kind (for example, their spectra are contained in the positive reals), the moments of xy are also moments of some measure, and this measure is called the multiplicative free convolution of μ and ν. (In general, since xy need not be self-adjoint, the moments of xy may not correspond to moments of any positive measure.) The use of word convolution is justified, because ordinary convolution of measures can be defined in a similar fashion, using ordinary independence. Multiplicative and additive free convolution, for which there is an analytical theory ([12], [13]) (which means they are actually computable), are very useful in finding spectra of large random matrices, given Theorem 1.1 and its analogs. Moreover, they appear to give tools to deal with limits of combinatorics of decomposition problems in representation theory of $U(n)$ ([2]).

In the following result of Speicher we use the notation $\mu \boxplus \nu$ for the free additive convolution of measures μ and ν:

Theorem 1.2. [10] *For a non-commutative self-adjoint random variable A, let μ_A to be its distribution, viewed as a measure on the real line. Let A_n and B_n be two sequences of self-adjoint $n \times n$ matrices, such that $\mu_{A_n} \to \mu_A$ and $\mu_{B_n} \to \mu_B$ in moments, for some compactly supported probability measures μ_A and μ_B on the reals. Then for almost all sequences of unitaries U_n, we have*

$$\mu_{A_n + U_n^* B_n U_n} \to \mu_A \boxplus \mu_B$$

in moments.

This should be interpreted as saying that the measure of the set of $n \times n$ unitary matrices $\mathcal{U}$, for which a specific moment of

$$\mu_{A_n + U^* B_n U}$$

differs by more than ϵ from the corresponding moment of

$$\mu_A \boxplus \mu_B,$$

for all $U \in \mathcal{U}$, goes to zero in the Haar measure as $n \to \infty$. We shall show how concentration phenomena allow to re-derive this result from Theorem 1.1 in the next section.

Freeness proves to be a useful concept for random matrices. But also random matrices have (and will) play a major role in the development of free probability theory and its applications to von Neumann algebras (see [16] and references therein; also [15])

[3] For example, take the identity function $f : x \mapsto x$ on the supports of μ, and a similar function g on the support of ν. These functions both generate commutative von Neumann algebras, and the measures define traces. Treated as non-commutative random variables, f and g have distributions μ and ν. Taking the free product of these two algebras provides us with two free random variables with desired distributions. The joint distribution of a free family does not depend on the "realization" of this family, so it does not matter in which way we construct those two free variables.

2. Concentration phenomena.

This section is based on [14].

2.1. **Levy families.** Let X_i be a sequence of metric spaces, each endowed with a Borel measure μ_i, with $\mu_i(X_i) = 1$. For a subset A of some X_i, by $N_\epsilon(A)$ we denote the ϵ-neighborhood of A in X_i.

Definition 2.1. [7] *A family $\{X_i, \mu_i\}$ is called* Levy, *if for any sequence of Borel sets $A_i \subset X_i$, such that* $\liminf_i \mu_i(A_i) > 0$, *and any ϵ, we have*

$$\lim_{i\to\infty} \mu_i(N_\epsilon(A_i)) \to 1.$$

Notice that X_i's need not have anything in common; this definition is just a useful property of a certain sequence of spaces. Roughly, being Levy means that for a fixed ϵ, ϵ-neighborhoods of sets of a nondecreasing measure become more and more dense in the whole space.

2.2. **Unitary random matrices.** As an example, consider random unitary matrices. We shall see in the next section that unitary groups with their Haar measure and the bi-invariant metric form a Levy family; assuming this result for the moment, we make the following computation. Let M_n denote the product of p copies of the Lie group of $n \times n$ unitary matrices $U(n)$. Define a metric space structure on M_n by setting the distance function

$$d_n(U_1, \dots, U_p, V_1, \dots, V_p) = \left(\mathrm{Tr}\Big(\sum_j (U_j - V_j)^*(U_j - V_j)\Big)\right)^{1/2} \tag{2}$$

where Tr denotes the trace which is n on the identity. This can be rewritten as

$$\sqrt{n}\left(\tau\Big(\sum_j (U_j - V_j)^*(U_j - V_j)\Big)\right)^{1/2}.$$

Here τ is the normalized trace, having value 1 on the identity element. This metric is equivalent to the metric obtained for a bi-invariant Riemannian metric, coming from the inner product on the Lie algebra given by

$$g(X_1, \dots, X_p, X_1, \dots, X_p) = -\sum_j \mathrm{Tr} X_j^2.$$

Thus with this metric and the Haar measure, our assumption implies that the family M_n is Levy.

Theorem 2.1. [14] *Let $\epsilon > 0$, $g = g_{i_1}^{k_1} \dots g_{i_m}^{k_m}$ be a reduced word in generators of the free group $\mathbb{F}(p)$ with generators $g_1, \dots g_p$, μ_n the Haar measure on $U(n)$, and τ_n the normalized trace, which is 1 at the identity matrix. Then for*

$$\Omega(\epsilon, g) = \{U_1, \dots, U_p \in U(n)^{\times p} : |\tau_n(U_{i_1}^{k_1} \dots U_{i_m}^{k_m})| < \epsilon\}$$

we have $\mu_n(\Omega(\epsilon, g)) \to 1$ as $n \to \infty$.

Proof. From (1.1), we know that

$$\int_{M_n} \tau_n(U_{i_1}^{k_1} \dots U_{i_m}^{k_m}) \to 0$$

as $n \to \infty$, because in the limit this expression is precisely a non-commutative moment of a free family of "Haar unitaries", and so vanishes by equation (1). The set Ω is then all unitaries that "almost" have the moments one would expect. The theorem says that as n increases, almost all unitaries have moments within ϵ of the expected limit.

Let now $f_n : M_n \to \mathbb{R}$ be defined by

$$f_n(U_1, \ldots, U_p) = \sqrt{n}\mathrm{Re}(\tau_n(U_{i_1}^{k_1} \ldots U_{i_m}^{k_m})).$$

It is easily seen that f is Lipschitz for the metric d_n, with constant $k = \sum k_i$ (which were the exponents in the word g.) Notice that the Lipschitz constant of f_n is independent of n.

Let now

$$\Xi(\delta, n) = \{(U_1, \ldots, U_p) : |f_n(U_1, \ldots, U_p)| \geq \delta\}.$$

If it were the case that

$$\liminf_n \mu_n\left(\Xi\left(\epsilon\sqrt{n}, n\right)\right) > 0,$$

then by the Levy property, we have that

$$\mu_n\left(\Xi\left(\frac{\epsilon\sqrt{n}}{2}, n\right)\right) \geq \mu_n\left(N_{\frac{\epsilon\sqrt{n}}{2k}}(\Xi(\epsilon\sqrt{n}, n))\right) \to 1.$$

(The factors of $\sqrt{n}$ account for the fact that we used the normalized trace in the definition of Ξ rather than the trace having value n on 1.)

But then

$$\liminf_n \frac{1}{\sqrt{n}} \int_{M_n} f_n d\mu_n \geq \liminf_n \frac{1}{\sqrt{n}} \frac{\epsilon\sqrt{n}}{2} > 0$$

since on $\Xi(\delta, n)$, f exceeds δ, which is a contradiction, because the integral must be zero in the limit. It follows that

$$\liminf_n \mu_n(\Xi(\epsilon\sqrt{n}, n)) = 0.$$

But the same reasoning applies to any subsequence. So the limit is zero.

Now exactly the same reasoning applied to the imaginary part of the trace will lead to the desired conclusion. □

It is obvious how to modify this theorem to also include constant diagonal matrices. Moreover, the result of Speicher that we stated as Theorem 1.2, is now an easy consequence (the only observation needed is that if U is a Haar unitary and is free from both A and B, then A and U^*BU are free; moreover, the additive free convolution is defined by

$$\mu_{x+y} = \mu_x \boxplus \mu_y$$

in the notation of Speicher, for x and y free. Furthermore, it is known ([11]) that moments of the free convolution of two measures are polynomials in the moments of the measures. Now it just remains to compare moments in Speicher's theorem, and apply the preceding theorem.)

Orthogonal groups $SO(n)$ also form a Levy family. The same proof works there to yield an analog of Theorem 2.1.

3. Isoperimetric Inequality and Concentration Phenomena

It remains for Theorem 2.1 to show that the products of unitary groups with the metric we defined in equation (2) and the Haar measure form a Levy family; we also need a similar result for the orthogonal groups.

3.1. **Curvature and Levy families.** In the sequel, for brevity, by a Riemannian manifold we shall always mean a compact manifold with no boundary endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$.

A principal example of a Levy family are spheres. Let X_i be the Euclidean sphere of radius r_i inside $\mathbb{R}^{i+1}$. Then for μ_i equal to the normalized volume measure on X_i, the family $\{X_i, \mu_i\}$ is Levy, if and only if

$$r_i i^{-1/2} \to 0$$

([8]).

The isoperimetric inequality allows one to reduce the general case of Levy families to spheres. For this case, the isoperimetric inequality (which in general gives an inequality for the area of the boundary of a domain once its volume is given) from [6] states that for any measurable $A \subset X$, and for any $\epsilon > 0$, we have that

$$\mu(N_\epsilon(A)) \geq \mu_k(N_\epsilon(B)),$$

where B is a ball on the sphere S^k of radius r, picked so that the Ricci curvature

$$R(S^k) = R(X),$$

where

$$R = \inf_\tau \mathrm{Ric}(\tau, \tau)$$

as τ runs over the whole of the unit tangent bundle (this is the infimum of the Ricci curvature; for a discussion, see [3]), and μ_k is the normalized measure on S^k and $k = \dim(X)$. The ball in chosen such that $\mu_k(B) = \mu(A)$.

Theorem 3.1. [7] *Let $\{X_i\}$ be closed Riemannian manifolds such that $R(X_i) \to +\infty$. Then the family $\{X_i, \mu_i\}$ is Levy.*

Here μ_i, as usual, denotes the normalized volume measure on the manifolds.

Proof. Suppose $A_i \subset X_i$. Then by what was said above, $N_\epsilon(\mu_i(A_i)) \geq$ measure of a certain ball on a certain sphere. Thus we are reduced to the case of the spheres. The spheres we pick have Ricci curvature $R(X_k)$; for a k-dimensional sphere of radius r, R is $(k-1)/r^2$, so the spheres are of radii $(k-1)^{1/2}/R(X_k)$, and so these spheres form a Levy family. □

As a corollary, using the known value for $R(SO(n)) = n/4$ ([7]), we find that $SO(n)$ form a Levy family.

3.2. λ_1 **and Levy families.** The next theorem will be useful in the case of unitary groups. Form M a Riemannian manifold, let $\lambda_1 = \lambda_1(M)$ be the first nontrivial eigenvalue of the Laplace operator $-\Delta$ (see Appendix.)

Theorem 3.2. [7] *If X_n are compact Riemannian manifolds and $\lambda_1(X_n) \to \infty$ then the family (X_n, μ_n) is Levy, where μ_n is the normalized measure element on X_n.*

Proof. Follows from the next theorem. □

Theorem 3.3. [7] *Let M be a compact Riemannian manifold. Let $A \subset M$ be a closed set with $\mu(A) = a > 0$, where μ is the normalized Riemannian volume measure on M. Then for $\epsilon > 0$,*

$$\mu(N_\epsilon(A)) \geq 1 - (1-a^2)\exp(-\epsilon\sqrt{\lambda_1(M)}\ln(1+a)).$$

The proof relies on an estimate of λ_1 coming from equation (5) (see Appendix) and can be found in [7].

3.3. Levy property for the unitary group.

We are now in a position to prove that $U(n)$ with their bi-invariant metrics and Haar measure form a Levy family. This presentation is based on [14].

Proposition 3.1. *$U(n)$ with their bi-invariant metric and Haar measure form a Levy family.*

Proof. It is known (see e.g., [17]), that for $U(n)$, the Laplacian is the operator on $L^2(U(n))$, given by $d\lambda(C)$, where λ is the left regular representation on $L^2(U(n))$, $d\lambda$ is the associated representation of the (complexified) Lie algebra, and C is the Casimir operator given by $\sum e_{ij}e_{ji}$, where e_{ij} are matrix units for the complexified Lie algebra (which is all of the complex matrices.)

If π is an irreducible representation of $U(n)$ with highest weight $(m_1 \leq \cdots \leq m_n)$, $d\pi$ the associated Lie algebra representation, then it is known (see [17]) that

$$d\pi(C) = \gamma_\pi 1$$

where

$$\gamma_\pi = \sum_{j=1}^{n} \ell_j^2 - (n-1)\sum_{j=1}^{n} \ell_j + \frac{n(n-1)(n-2)}{6},$$

where $\ell_j = m_j + n - j$. To rewrite this,

$$\gamma_\pi = \sum_{j=1}^{n} \left(\ell_j - \frac{n-1}{2}\right)^2 - \frac{n(n-1)^2}{2} + \frac{n(n-1)(n-2)}{6}.$$

Thus for any $a > 0$, for sufficiently large n, and any π with nontrivial weights $(0 \leq m_1 \leq m_2 \leq \cdots \leq m_n \neq 0)$, we have that $\gamma_\pi > a$. But by the Peter-Weyl theorem,

$$\lambda = \sum_{\pi} \pi \otimes 1_{\dim(\pi)}$$

where the sum is over all irreducible representations π. Therefore, in this picture, the Laplacian is a diagonal operator, with eigenvalues γ_π for various π. But then from what we said, as $n \to +\infty$ the smallest nontrivial eigenvalue goes to $+\infty$ as well.

Therefore, by Theorem 3.2, we have that $U(n)$ form a Levy family. □

It is not difficult to see from equation (5) that for $U(n)^{\times p}$, the value of λ_1 is at least as big as p times its value for $U(n)$. In fact, it is p times the value for $U(n)$.

4. Appendix: the Laplacian.

This section gathers some results about the Laplace operator needed in the sequel. Most of these results are from [5].

4.1. **Definitions.** Suppose that M is a Riemannian manifold. Then the metric defines an isomorphism of its cotangent bundle T^*M and its tangent bundle TM. In particular, this means that to any function f one can associate a vector field, denoted ∇f, by taking the vector field associated to the differential df by the above isomorphism. In other words, if ξ is a tangent vector, then ∇f is determined by

$$\langle \nabla f, \xi \rangle = df(\xi) = \xi f. \tag{3}$$

Moreover, given a vector field X, we have an associated linear mapping $TM \to TM$, given by

$$\nabla X : \xi \mapsto \nabla_\xi X$$

Here ∇_ξ stands for covariant differentiation with respect to ξ (unique such compatible with the Riemannian metric and torsion-free [4].) But we have a natural trace on the algebra of all linear mappings from a single fiber to itself; therefore,

$$\mathrm{div} X = \mathrm{Tr}(\xi \mapsto \nabla_\xi X) \tag{4}$$

defines a real-valued function on M, called the divergence of X.

Combining the two definitions, we have

Definition 4.1. *The negative of the* Laplacian *is defined to be the closed linear differential operator* $\Delta : f \mapsto \mathrm{div}\nabla f$ *acting on a dense subspace of* $L^2(M)$.

Here $L^2(M)$ stands for the Hilbert space of all square-integrable functions f (with inner product $(f,g) = \int_M fg dV$). The measure with respect to which the integral is taken is the Riemannian volume on M; its density in a neighborhood system is given by

$$dV = \sqrt{g} dx^1 \dots dx^n,$$

here $\sqrt{g}$ is defined to be the square root of the determinant of the matrix of g in our coordinate system.

In the case of the Euclidean space, definitions above reduce to the usual notions of gradient, divergence, and Laplace operator.

4.2. **Spectral properties.** Like in the Euclidean case, the Laplace operator is always unbounded. Its spectrum is always real as it is self-adjoint. From the fact that ∇ and div are adjoints of one another with respect to $(\cdot,\cdot)$ on $L^2(M)$ and the inner product on vector fields on M, defined by

$$(X,Y) = \int_M \langle X(p), Y(p) \rangle_p dV(p),$$

we know that the spectrum of $-\Delta$ is nonnegative. Moreover, it is also easy to see that constants span the eigenspace for 0. However, much more is known about its spectral distribution:

Theorem 4.1. [5] *The spectrum of* $-\Delta$ *is discrete and positive, consisting of numbers* $0 < \lambda_1 < \lambda_2 < \dots$ *increasing to infinity; moreover, each associated eigenspace is finite-dimensional. In fact, those eigenspaces are spanned by* C^∞ *functions.*

The proof of this fact is rather involved, and the reader is referred to [1], page 250, for an exposition. The main point of the argument is to show that the operator

$$(\Delta - \lambda)^{-1}$$

is compact for non-real λ, and so has a discrete spectrum, having only a possible accumulation point at zero and its spectral projections are of course compact, so finite-dimensional (which implies that the spectrum of Δ is discrete, perhaps "accumulating" at infinity, and also that eigenspaces have finite dimension.)

There is a formula ([7]) for the first eigenvalue λ_1: it is the largest constant for which

$$\lambda_1 \|f\|_{L_2}^2 \leq (-\Delta f, f) = \int_M \|\nabla f\|^2 \tag{5}$$

for any continuous sufficiently smooth function f for which $\int_M f = 0$.

This formula is not difficult to prove. First, we have

$$\Delta f = 0$$

identically implies that f is constant. Let C be the (one-dimensional) subspace of $L^2(M)$ spanned by the constants. Then λ_1 is the largest constant for which

$$\lambda_1 \leq \frac{(-\Delta f, f)}{\|f\|^2}$$

for all $f \in C^\perp$. But $C^\perp$ exactly consists of those functions whose integral over M vanishes.

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