THE MASLOV CLASS AND THE SECOND FUNDAMENTAL FORM
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ABSTRACT. The Maslov class is an important cohomological invariant which appears naturally in the resolution by the Hamilton-Jacobi method of the Shrödinger equation of quantum physics. It was discovered by V.P. Maslov [7] in 1965, but it was V.I. Arnold [1] who first studied it in detail and characterized it as an obstruction to the transversality of a Lagrangian submanifold of a cotangent bundle, with the vertical subbundle. In many cases it has been found possible to express the Maslov class solely in terms of the Riemannian structure of the Lagrangian immersion, associated to the Kähler metric on the ambient symplectic manifold. This paper is a survey of some of the work done in this direction.

1. Definitions and Preliminaries

Throughout this paper the Einstein summation convention is assumed unless stated otherwise.

We start with

Definition 1.1. *Let V be a real vector space. A* symplectic structure *on V is a skew-symmetric nondegenerate bilinear form $\omega : V \times V \to \mathbb{R}$.*

The nondegeneracy condition implies that the dimension of V must be even.

Example 1.2. $V = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ *with*

$$\omega = q^i \wedge p_i \tag{1}$$

where q^i and p_i are the canonical coordinates on $\mathbb{R}^n \times \mathbb{R}^n$, is a symplectic vector space. This linear symplectic structure is called standard*, or* canonical*.*

It can be shown, by the usual normalization argument, that any symplectic vector space (V, ω) admits a basis (called *symplectic*) with respect to which ω is given by (1) (cf.[2]).

Given a subspace W of a symplectic space V we define its *symplectic orthogonal*, denoted $W^\dagger$, to be the complimentary subspace to W with respect to ω, i.e. $\omega(W, W^\dagger) = 0$.

W is said to be *isotropic* if $W \subseteq W^\dagger$, *coisotropic* if $W^\dagger \subseteq W$, *Lagrangian* if it is both isotropic and coisotropic. The nondegeneracy of ω then implies that any Lagrangian subspace has dimension equal to half the dimension of V, and moreover, every isotropic subspace is contained in a Lagrangian subspace, and every coisotropic subspace contains one.

For instance, every one-dimensional subspace is isotropic, whereas every subspace of codimension one is coisotropic, but in general not every subspace of middle dimension is Lagrangian.

An endomorphism $A : V \to V$ satisfying $\omega(Ax, Ay) = \omega(x, y)\ \forall x, y \in V$ is called a *linear symplectomorphism*, and such endomorphisms form a subgroup of $GL(V)$, denoted $Sp(V)$.

Given a symlectic vector space (V, ω), an *ω-compatible complex structure* is a symplectomorphism $\mathbb{J} : V \to V$ such that $\mathbb{J}^2 = -1$ and $g = \langle\cdot,\cdot\rangle = \omega(\cdot, \mathbb{J}\cdot)$ defines a positive-definite inner product on V.

In terms of $\mathbb{J}$, any isotropic subspace $W \subseteq V$ is characterized by the property that $\mathbb{J}W \perp W$ with respect to g. In contrast, a subspace $M \subseteq V$ is called *complex* (with respect to $\mathbb{J}$) if $\mathbb{J}M \subseteq M$. Thus, we have the following direct sum decomposition:

$$W^{\perp} = \mathbb{J}W \oplus M \tag{2}$$

where M is the maximal complex subspace of $W^{\perp}$. Evidently, W is Lagrangian iff $M = \{0\}$.

It can be easily shown (cf [12]) that every symplectic vector space admits a compatible complex structure and the set of such structures is contractible in $Sp(V)$.

Definition 1.3. *The collection of all Lagrangian subspaces of a linear symplectic space (V, ω) is called the* Lagrangian Grassmanian *of V, denoted by $\mathcal{L}(V)$.*

Choose a symplectic basis in V, thereby identifying it with $\mathbb{R}^{2n}$, and then consider the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$. Then the following holds:

Lemma 1.4. [1] *The unitary group $U(n)$ acts transitively on $\mathcal{L}(n)$, and the isotropy subgroup of the real subspace $\mathbb{R}^n$ is $O(n)$. Hence, $\mathcal{L}(n) \cong U(n)/O(n)$.*

The square of the determinant $\det^2 : U(n) \to S^1$ factors through $U(n)/O(n)$ and hence, induces a map

$$\det{}^2 : \mathcal{L}(n) \to S^1$$

The inverse image of 1 under this map is $SU(n)/SO(n)$, and so by considering the appropriate exact sequence of homotopy groups one finds that $\det^2$ induces an isomorphism of fundamental groups (cf.[1]).

Let α be the fundamental cohomological class of S^1, represented by the 1-form $\frac{dz}{2\pi i z}$ on $S^1 \subseteq \mathbb{C}$.

Definition 1.5. *The cohomology class $\mu \in H^1(\mathcal{L}(n), \mathbb{Z})$ given by*

$$\mu = (det^2)^* \alpha$$

is called the universal Maslov class*.*

It can be shown that this definition is in fact independent of the choice of the symplectic basis and the complex structure on V [12].

Definition 1.6. *A* symplectic manifold *is a smooth manifold P endowed with a closed nondegenerate 2-form ω.*

Thus, the tangent space to each point of a symplectic manifold is a symplectic vector space, making TP a *symplectic vector bundle.*

Example 1.7. $P = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$; $\omega = dq^i \wedge dp_i$. *This symplectic structure is called* canonical.

Example 1.8. $P = T^*M$; $\omega = -d\theta$, *where* θ *denotes the* Liouville 1-form *given by:*

$$\theta(\beta)(v) = \beta(\pi_* v),$$

where $\beta \in T^*M$, $v \in T_\beta(T^*M)$, $\pi : T^*M \longrightarrow M$ *is the canonical projection.*

Example 1.7 is a special case of Example 1.8. In fact, a theorem of Darboux-Weinstein ([12]) asserts that any symplectic manifold is locally symplectomorphic to the P of Example 1.7.

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Definition 1.9. *A submanifold* I *of a symplectic manifold* (P, ω) *is called* isotropic *(resp.* Lagrangian*) if* T_pI *is an isotropic (resp. lagrangian) subspace of* T_pP $\forall p \in \ I \subseteq \ P$.

More generally, we can define isotropic and Lagrangian immersions. Thus, an immersion

$$\iota : I \longrightarrow P$$

is isotropic if $\iota^*\omega \equiv 0$, and it is Lagrangian if also $\dim P = 2 \dim I$.

For instance, any 1-form $\phi : M \longrightarrow T^*M$ defines a projectable embedding which is Lagrangian iff ϕ is closed. If $\phi = df$, f is called a *phase function.* Such phase functions appear in approximate solutions to the Schrödinger equation, and it has been found necessary to generalize the concept of a phase function to Lagrangian embeddings which are not exact or projectable. It turns out that the possibility of doing so depends on the Liouville class of the embedding (i.e., the cohomology class of the pullback of the Liouville form on T^*M), and its Maslov class, which we will presently define (see [12] for details).

2. The Maslov class - The Euclidean case.

Consider $\mathbb{R}^{2n}$ with the standard Euclidean metric, standard symplectic structure and standard complex structure.

Let $\iota : L \longrightarrow \mathbb{R}^{2n}$ be a Lagrangian immersion. Then we define the *Gauss map* by:

$$G_\iota : L \longrightarrow \mathcal{L}(n) \tag{3}$$

by $m \mapsto \iota_*(T_mL) \in \mathcal{L}(n)$, where $T_m\mathbb{R}^{2n}$ is naturally identified with $\mathbb{R}^{2n}$

Definition 2.1. *The* Maslov class *of a lagrangian immersion* $\iota : L \longrightarrow \mathbb{R}^{2n}$ *is defined by*

$$\mu_\iota = G_\iota^* \mu \in H^1(L, \mathbb{Z}),$$

where μ *is the universal Maslov class (Def. 1.5).*

Given any homology class $[\gamma] \in H_1(L, \mathbb{Z})$ represented by a cycle $\gamma : S^1 \to L$, we then define the *Maslov index* of γ to be $\mu_\iota(\gamma) \in \mathbb{Z}$.

At each $m = (q_0, p_0) \in \mathbb{R}^{2n}$ we define the *vertical space* V_m to be the subspace of $T_m\mathbb{R}^{2n}$ consisting of the vectors tangent to the vertical fibre $\{q = q_0\}$. It is evident that V_m is a Lagrangian subspace of $T_m\mathbb{R}^{2n}$.

In 1967 V.I. Arnold proved the following

Theorem 2.2. [1] *The Maslov index of* $[\gamma] \in H_1(L, \mathbb{Z})$ *equals the intersection number of* γ *with the* $(n-1)$*-cycle* $\Sigma = \{m \in L | \, V_{\iota(m)} \cap \iota_*(T_m L) \neq \{0\}\}$.

Thus, the Maslov class μ_ι represents an obstruction to the transversality of two Lagrangian subbundles of $\iota^*(T\mathbb{R}^{2n})$, TL and the vertical bundle $\iota^* V$.

Example 2.3. *The embedded circle* $S^1 \hookrightarrow \mathbb{C}^1$ *has Maslov index 2, while the Maslov index of the figure-eight* ($\iota(t) = (\cos t, \sin 2t) \, t \in \mathbb{R}$) *is zero. In general, the Maslov index of a curve in* $\mathbb{C}^1$ *equals twice its rotation index.*

In 1981 J.M. Morvan gave the following description of the Maslov class of a Lagrangian immersion $\iota : L \to \mathbb{R}^{2n}$ in terms of Riemmanian invariants of the immersion.

Denote by $\langle \cdot, \cdot \rangle$ the induced Riemannian metric on L, and let ∇ be the Levi-Civita connection on L, $\tilde{\nabla}$ the trivial connection on $\mathbb{R}^{2n}$, and consider the second fundamental form $\sigma : TL \times TL \to T^\perp L$ of the immersion, given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y \tag{4}$$

and the mean curvature vector field

$$H = \frac{1}{n} \mathrm{Tr} \sigma \tag{5}$$

Recall that L is called *minimal* if $H \equiv 0$.

Morvan proved the following

Theorem 2.4. *([9]) The Maslov class of a Lagrangian immersion* $\iota : L \to \mathbb{R}^{2n}$ *is given by*

$$\mu_\iota = \frac{n}{\pi} \left[i_H \omega_{|TL} \right] ,$$

where i_H *denotes the interior product.*

Sketch of proof. The proof of this theorem is based on the following observations: first, just as is the case with any Grassman manifold, $T_\lambda \mathcal{L}(n)$ can be identified with a certain subspace of $\hom(\lambda, \lambda^\perp)$, in this case the one consisting of linear maps L satisfying

$$\langle L(X), \mathbb{J}(Y) \rangle = \langle L(Y), \mathbb{J}(X) \rangle \quad \forall X, Y \in \lambda.$$

In particular, $\mathbb{J} : \lambda \to \lambda^\perp$ is such a map, and therefore, $\mathbb{J}$ can be regarded as a vector field on $\mathcal{L}(n)$.

Using the natural metric on the Grassmanian, we can associate to $\mathbb{J}$ a 1-form $\tilde{\mathbb{J}}$ on $\mathcal{L}(n)$. Then, using the fact that $T(\det^2)(Z) = 2\mathrm{Tr}(Z)$, $\forall Z \in \mathfrak{u}(n)/\mathfrak{o}(n)$, one can show by a straightforward computation that

$$(\det{}^2)^* \left(\frac{dz}{2\pi i z} \right) = \frac{1}{\pi} \tilde{\mathbb{J}},$$

thus expressing the universal Maslov class in terms of $\tilde{\mathbb{J}}$. (see Def. 1.5) Then, it is a known fact (see [4], for instance) that

$$TG_\iota : TL \to T\mathcal{L}(n) \subset T^*L \otimes T^\perp L$$

equals the second fundamental form when the latter is regarded as a map:

$$\tilde{\sigma} : TL \to T^*L \otimes T^\perp L.$$

Using this and (7), another computation shows that

$$G_\iota^* \tilde{\mathbb{J}} = n i_H \omega.$$

The theorem follows immediately from this. ∎

3. Kähler Manifolds

Let P be a smooth manifold with a Riemannian metric $\langle\cdot,\cdot\rangle$, admitting an almost complex structure $\mathbb{J} \in \Gamma(TP \otimes T^*P)$ (i.e., $\forall p \in P$, $\mathbb{J}_p : T_pP \to T_pP$ is an isometry with $\mathbb{J}_p^2 = -\mathrm{Id}_{T_pP}$). Then

$$\omega(X,Y) = \langle \mathbb{J}X, Y\rangle \tag{6}$$

defines a nondegenerate 2-form on P. If $d\omega = 0$, P is called *almost Kähler*.

Thus, any almost Kähler manifold is symplectic by definition. Conversely, any symplectic manifold (P,ω) admits an ω-compatible almost complex structure $\mathbb{J}$ ([12]) making it into an almost Kähler manifold with the metric given by

$$\langle X, Y\rangle = \omega(X, \mathbb{J}Y) \tag{7}$$

It's easy to see that the condition $d\omega = 0$ is equivalent to $\nabla\mathbb{J} = 0$ where ∇ denotes the Levi-Civita connection associated to $\langle\cdot,\cdot\rangle$.

An almost Kähler manifold is called *Kähler* if the almost complex structure $\mathbb{J}$ is induced by a complex structure on P (i.e. P is a complex manifold and $\mathbb{J}$ denotes multiplication by $\sqrt{-1}$ in the fibres of TP)

Let P be an almost Kähler manifold, and let

$$\iota : L \to P$$

be a Lagrangian immersion. Let $\tilde{\nabla}$ be the levi-Civita connection for the Kähler metric on P, ∇ the induced connection on L, σ the second fundamental form of the immersion, and H the mean curvature vector field along L ((4),(5)). Choose a local field of orthonormal frames

$$e_1, \ldots, e_n, \quad e_{1^*} = \mathbb{J}e_1, \ldots, e_{n^*} = \mathbb{J}e_n$$

in P such that, restricted to L, $e_1, \ldots, e_n$ are tangent to L (recall (2)). Let $\tilde{R}$ be the curvature tensor of $\tilde{\nabla}$, $\tilde{S}$ the *Ricci tensor* defined by

$$\tilde{S}(X,Y) = \mathrm{Tr}(Z \mapsto \tilde{R}(X,Z)Y) \tag{8}$$

or, locally,

$$\tilde{S}_{\alpha\beta} = \sum_\gamma \tilde{R}^\gamma_{\alpha\gamma\beta}, \quad \alpha,\beta,\gamma = 1,\ldots,n^* \tag{9}$$

Define the following 1-forms on L:

$$\alpha_H = i_H\omega_{|TL} \tag{10}$$

$$\Theta = \sum_{i=1}^{n} \phi_i^{i^*}, \tag{11}$$

where ϕ is the pulled-down connection form of $\tilde{\nabla}$, restricted to L. Then we have:

$$\begin{aligned}
\Theta(e_j) &= \phi_i^{i^*}(e_j) = \langle \tilde{\nabla}_{e_j} e_i, e_{i^*}\rangle = \langle \tilde{\nabla}_{e_j} e_i - \nabla_{e_j} e_i, e_{i^*}\rangle \\
&= \langle \sigma(e_j, e_i), e_{i^*}\rangle = \langle \sigma(e_i, e_j), e_{i^*}\rangle = \langle \sigma(e_i, e_i), e_{j^*}\rangle \\
&= \langle nH, \mathbb{J}e_j\rangle = -n\langle \mathbb{J}H, e_j\rangle = -\omega(H, e_j) = -n\alpha_H(e_j)
\end{aligned}$$

Thus, we have the following simple relation between Θ and α_H:

$$\Theta = -n\alpha_H \tag{12}$$

Using the structure equations of the immersion, a straightforward computation yields the following result, due to P.Dazord ([10], [5]):

$$d\alpha_H = \frac{1}{n}\iota^*\rho \tag{13}$$

where $\rho(X, Y) = \tilde{S}(\mathbb{J}X, Y)$ is the Ricci form of P.

Thus, in general, α_H (and hence, Θ) need not be closed, so no matter how we end up defining the Maslov class of L, we cannot expect Theorem 2.4 to hold in this general situation. However, there is an important class of Kähler manifolds, called *Einstein-Kähler*, for which α_H is closed.

Definition 3.1. *A Riemannian manifold (M^n, g) is called* Einsteinian *if $\exists f \in C^\infty(M)$ such that $S = fg$, where S denotes the Ricci tensor.*

It can be shown that in fact, for $n \geq 3$, f must be constant (see [3]).

So, if P is Einstein-Kähler, $\tilde{S} = f\langle\cdot,\cdot\rangle$, hence $\rho = f\omega$, and so by (13) α_H is closed since L is Lagrangian.

We are now ready to give a general definition of the Maslov class.

4. Constructions in cotangent bundles.

Suppose $V = M \times \mathbb{R}^{2n}$ is a trivial symplectic bundle over M (the trivialization is fixed). Then any Lagrangian subbundle E of V induces a map

$$G_E : M \to \mathcal{L}(n), \tag{14}$$

given by $m \mapsto E_m$. We can then use G_E to pull back the universal Maslov class, but the result will in general depend on the choice of trivialization. However, if we choose a *reference* Lagrangian subbundle E_0, then

$$\mu(E, E_0) = (G_E^* - G_{E_0}^*)\mu \tag{15}$$

where μ is the universal Maslov Class defined in Sec. 1, is well-defined and called the *Maslov class of E with respect to E_0*. It is an obstruction to the transversality of E with E_0 ([12]).

For nontrivial V, note that since $Sp(n)$ is connected, it follows that for any loop $\gamma : S^1 \to M$ the pullback bundle $\gamma^* V$ is trivial. Thus $\mu(E, E_0)$ is well-defined by the requirement that for any smooth loop γ in M,

$$\gamma^* \mu(E, E_0) = \mu(\gamma^* E, \gamma^* E_0).$$

Definition 4.1. *Let $P = T^* M$, with the canonical symplectic structure ω (Example 1.8), and let $\iota : L \to T^* M$ be a Lagrangian immersion. Then the* Maslov class *of (L, ι) is, by definition,*

$$\mu_\iota = \mu(TL, \iota^* VM) \tag{16}$$

where TL and $\iota^* VM$ are Lagrangian subbundles of the symplectic bundle $\iota^* T(T^* M)$ over L, $V = VM$ denotes the vertical subbundle (i.e., given by the condition $\pi_* V = 0$, where $\pi : T^* M \to M$ denotes the canonical projection).

For $M = \mathbb{R}^n$ this gives back Def. 2.1 as $G_{\iota^* V}$ is a constant map.

As any symplectic manifold, $T^* M$ admits a Kähler metric, but formula (13) prevents one from directly generalizing Theorem 2.4 to arbitrary cotangent bundles. However, if one is careful in choosing the metric and is willing to consider connections other than Levi-Civita, this can be done.

We start with an arbitrary Riemannian metric g on M. The Levi-Civita connection of g on the bundle of frames $F(M)$ induces a connection on $T^* M$, and gives a *horizontal distribution* on $T(T^* M)$. It's clear that

$$T(T^* M) = H \oplus V,$$

where V is the vertical subbundle. Now we can define a metric $\hat{g}$ as follows:

$$\hat{g}_p(v, w) = \begin{cases} g(\pi_* v, \pi_* w) & \text{if } v, w \in H_p \\ 0 & \text{if } v \in H_p,\ w \in V_p, \text{ or v.v.} \\ \tilde{g}(v, w) & \text{if } v, w \in V_p \end{cases}$$

Here $\tilde{g}$ is the induced inner product in the fibres of $T^* M$ and V_p is identified with $T^*_{\pi(p)} M$.

It turns out that the horizontal subbundle H is Lagrangian (and so, of course, is V), and we have the following technical theorem:

Theorem 4.2. [11] *If (M, g) is a Riemannian manifold, then g induces a metric $\hat{g}$ on $T^* M$ such that $(T^* M, \hat{g}, \omega)$ becomes an almost Kähler manifold. This manifold is strictly Kähler iff g is flat.*

Furhtermore, there is a well-defined metric connection $\hat{\nabla}$ on $T^* M$ (i.e. $\hat{\nabla} \hat{g} = 0$), such that the $\hat{\nabla}$-parallel translations preserve H and V, and the torsion of $\hat{\nabla}$ has no horizontal component. $\hat{\nabla}$ and g are simultaneously flat, and then $\hat{\nabla}$ is precisely the Levi-Civita connection of the metric $\hat{g}$.

$\hat{\nabla}$ is called the *complex lift* of g to $T^* M$. It can be regarded as a generalization of the construction of $\mathbb{C}^n$ from $\mathbb{R}^n$. Even though $\hat{\nabla}$ is not in general a Levi-Civita connection, it can still be used, given a Lagrangian immersion $\iota : L \to T^* M$, to get the induced connection $\hat{\nabla}'$ on L, the second fundamental form $\hat{\sigma}$ (not necessarily symmetric), and the 1-form $\hat{\Theta}$, defined by (11). Notice, however, that because of the torsion, (12) no longer holds. Yet, $\hat{\Theta}$ turns out to be closed, while α_H is not, and moreover, the Maslov class can be expressed in terms of $\hat{\Theta}$:

Theorem 4.3. [11] *The Maslov class of a Lagrangian immersion $\iota : L \to T^*M$ is given by*

$$\mu_\iota = -\frac{1}{\pi}[\hat{\Theta}] \tag{17}$$

Corollary 4.4. *Let $\iota : L \to T^*M$ be a Lagrangian immersion, where M is a flat Riemannian manifold, and let H be the mean curvature vector field of L. Then*

$$\mu_\iota = \frac{n}{\pi}[\alpha_H] \tag{18}$$

Proof. Follows immediately from the previous two theorems and formula (12), which is valid in the case of a flat metric, by the last assertion of Theorem 4.2 and (13). ■

The above also works if we take P to be any Kähler manifold which happens to have a parallel Lagrangian distribution E_0, for then it follows that E_0 is integrable and P is flat (cf. [11]), and hence, locally isometric to $\mathbb{C}^n$ in such a way that E_0 goes over to the "purely imaginary" subspace. So, since the computation of the representative differential form of the Maslov class has a local character, Theorem 4.2 and Corollary 4.4 imply

Corollary 4.5. *Let (P, g, ω) be a Kähler manifold which admits a parallel Lagrangian distribution E_0, and let $\iota : L \to P$ be a Lagrangian immersion. Then the Maslov class $\mu(TL, \iota^*E_0)$ is given by (18).*

If the manifold P is Einstein-Kähler, then the above result can be generalized even further. In this case the form α_H is closed by (13), and we have the following remarkable theorem due to P. Dazord:

Theorem 4.6. [11] *Let (P, g, ω) be a simply connected Einstein-Kähler manifold, endowed with an arbitrary Lagrangian distribution E_0. Let $\iota : L \to P$ be a Lagrangian immersion with the mean curvature vector H. Then $\mu(TL, \iota^*E_0)$ is given by (18).*

[1]

Example 4.7. *Let $P = \mathbb{R}^2 - \{0\}$, with the canonical Kähler structure. Let E_0 be the Lagrangian foliation of P consisting of the concentric circles about the origin, and let L be one of those circles. Then $\mu(TL, E_0)$ is obviously zero, whereas (18) gives the usual value of twice the fundamental class of the circle, since removing the origin does not affect the Riemannian structure of the embedding. Thus the simple connectivity assumption in Theorem 4.6 is indispensable.*

Despite this example, one is motivated to make the following

[1]This theorem is stated and proved in [11] under the assumption that P is Ricci-flat; however, the proof only uses the restriction of the Ricci form to L and E_0, and so, since L and E_0 are Lagrangian, it suffices for P to be Einstein-Kähler.

Definition 4.8. *Let P^{2n} be an Einstein-Kähler manifold, and let $\iota : L \to P$ be a lagrangian immersion. Then the* Maslov class *of ι is defined by*

$$\mu_\iota = \frac{n}{\pi}[\alpha_H]$$

The next section relates the Maslov class to the issues of minimality.

5. Deformations.

First we briefly recall some notions from the Hodge-DeRham theory of harmonic differentials. The reader is referred to the appropriate literature for details (see, for example, [8] or [3]).

Let (M^n, g) be a compact oriented Riemannian manifold. Then the *Riemannian volume element* on M is the unique n-form ν_g which assigns the value 1 to every positively oriented orthonormal basis of T_xM $\forall x \in M$. Recall also that g naturally induces an inner product on any tensor bundle over M, which we also denote by g. Then for any p with $0 \le p \le n$ the *Hodge $*$ operator* is defined to be the unique vector bundle isomorphism

$$* : \Lambda^p(T^*M) \to \Lambda^{n-p}(T^*M)$$

such that

$$\alpha \wedge (*\beta) = g(\alpha, \beta)\nu_g \tag{19}$$

$\forall\, \alpha, \beta \in \Lambda^p(T_x^*M)$, $\forall\, x \in M$.

Then the global inner product on $\Omega^p(M)$ is given by

$$(\omega, \eta) = \int_M g(\omega, \eta)\nu_g = \int_M \omega \wedge *\eta \tag{20}$$

With respect to this inner product, the exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ has a formal adjoint, called the *codifferential* $\delta : \Omega^{p+1}(M) \to \Omega^p(M)$, given by

$$\delta = - * \circ d \circ *$$

and satisfying

$$(d\omega, \eta) = (\omega, \delta\eta) \tag{21}$$

$\forall\, \omega \in \Omega^p(M)$, $\eta \in \Omega^{p+1}(M)$.

The operator $\Delta = d\delta + \delta d : \Omega^p(M) \to \Omega^p(M)$ is the *Hodge-deRham Laplacian* on p-forms. A p-form ω is called *harmonic* if $\Delta\omega = 0$. A harmonic form is necessarily both closed and co-closed. The converse is obvious

Now let $(P^{2n}, \langle,\rangle, \omega)$ be an almost Kähler manifold, L a compact Lagrangian submanifold. Suppose ξ is a vector field on P defined in a neighborhood of L such that $\xi|_L$ is normal to L. The local flow ϕ_t of ξ gives rise to a normal variation of L in P. We consider *the first variation of volume*

$$V'(\xi) = \frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\phi_t(L))$$

It is known (see, for instance, ref [6]) that

$$V'(\xi) = -n \int_L \langle H, \xi \rangle \, d\nu \tag{22}$$

where H and ν are, respectively, the mean curvature vector and the Riemannian volume element of L. Given ξ, we define the one-form α_ξ by formula (11), with H replaced by ξ.

Given a Lagrangian embedding $\iota : L \hookrightarrow P$, we may only consider those normal variations ξ for which $\phi_t(L)$ stays Lagrangian for sufficiently small t. Then the infinitesimal version of this condition is given by

$$\iota^*(\mathcal{L}_\xi \omega) = 0$$

where $\mathcal{L}$ denotes the Lie derivative. By Cartan's magic formula, this is equivalent to

$$d\alpha_\xi = 0$$

Thus we call a normal variation ξ *Lagrangian* if α_ξ is closed. It is *Hamiltonian* (resp. *harmonic*) if α_ξ is exact (resp. harmonic).

Recall that a submanifold $L \subset P$ is called *minimal* if $V'(\xi) = 0$ for any normal variation ξ. By (22) this is equivalent to the condition $H \equiv 0$. We can generalize the notion of minimality by only considering Lagrangian, etc. variations

Definition 5.1. *A Lagrangian submanifold L of a Kähler manifold is said to be* isotropic minimal *(resp.* Hamiltonian-minimal *or* harmonic-minimal*) if $V'(\xi) = 0$ for any Lagrangian (resp. Hamiltonian or harmonic) variation ξ.*

We now derive the Euler-Lagrange equation for Hamiltonian- and harmonic minimal Lagrangian submanifolds.

Theorem 5.2. *Let P be a Kähler manifold. Then a Lagrangian submanifold $L \subset P$ is*

- **(a):** *Hamiltonian-minimal iff α_H is co-closed*
- **(b):** *Harmonic-minimal iff α_H is a sum of an exact one-form and a co-exact one-form,*

where, as usual, H denotes the mean curvature vector of L.

Proof. By the first variation formula (22) the minimality condition implies:

$$0 = \int_L \langle H, \xi \rangle \, d\nu = \int_L \langle \alpha_H, \alpha_\xi \rangle \, d\nu = (\alpha_H, \alpha_\xi)$$

where ξ is either Hamiltonian or harmonic. In the former case, by definition, $\alpha_\xi = df$ for some function f on L, and conversely, for any f we can find some ξ such that $\alpha_\xi = df$. Therefore, L is Hamiltonian-minimal iff

$$(df, \alpha_H) = (f, \delta\alpha_H) = 0$$

for all f, which is equivalent to

$$\delta\alpha_H = 0$$

Similarly, L is harmonic-minimal iff α_H is perpendicular to all harmonic one-forms on L. Thus, by the Hodge-deRham decomposition of one-forms (cf. [8]) we deduce the statement **(b)**. ∎

Corollary 5.3. *Let P be Einstein-Kähler. Then L is*

(a): *Hamiltonian-minimal iff α_H is harmonic*
(b): *Harmonic-minimal iff α_H is exact.*

Proof. Since P is Einstein-Kähler, (13) implies $d\alpha_H = 0$. So, if L is Hamiltonian-minimal, Theorem 5.2 implies that α_H is both closed and co-closed which is true iff it is harmonic; if L is harmonic-minimal, the condition $d\alpha_H = 0$ implies that α_H is perpendicular to every co-exact one-form, and then the same Theorem implies that α_H must be exact. ∎

Corollary 5.4. *A Lagrangian submanifold of an Einstein-Kähler manifold is harmonic minimal iff its Maslov class vanishes.*

Proof. Immediate from Def. 4.8 and the previous Corollary. ∎
All the results of this section and their generalizations can be found in [10] and [5].

References

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