Singular Riemannian Geometry

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1 Introduction

Suppose that M is a Riemannian manifold of topological dimension d . To define a **singular Riemannian metric** on M, one assigns a k-dimensional distribution to M, where $k < d$. This distribution, H, forms the new horizontal path space for M and thus limits the direction of geodesic flow. The length of vectors which lie in H is determined by the Riemannian metric, and vectors transverse to H are assumed to have infinite length. A geodesic path is said to be **admissible** if all its tangent vectors lie in H, and one defines the distance, $\rho_S(p,q)$, to be the infimum of lengths of admissible (piecewise smooth) paths from p to q.

Such structures have many interesting applications. They are utilized in the theory of nilpotent Lie groups **[Gr1,Gr2,M]**, ergodic theory (where they are used to compute topological entropy) **[Gr1]**, and in stochastic differential equations and optimal control theory **[B1,B2]**.

What follows is a general introduction to singular Riemannian geometry as well as some interesting results related to connectivity, completeness and Hausdorff dimension.

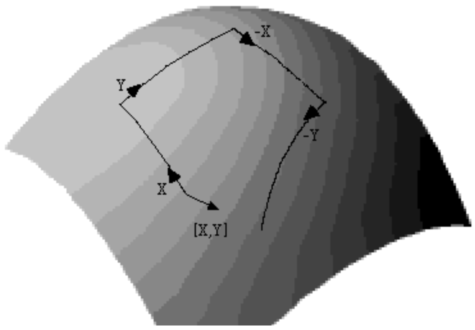
Example 1 Take $M = \mathbf{R}^3 - \{0\}$ with spherical coordinates and the Euclidean metric, ρ. Consider the singular Riemannian metric, ρ_S, induced by $X_1 = \frac{\partial}{\partial\theta}$ and $X_2 = \frac{\partial}{\partial\phi}$. Then a particle travelling along admissible paths never changes its radius, so that ρ_S is finite on each sphere about the origin, but the distance between points of different spheres is infinite. Thus, this metric induces a foliation of M into a continuum of disjoint 2-dimensional components. Here, the metric topology for (M,ρ_S) is different than that for (M,ρ), but this is not in general the case.

2 Connectivity

Let $X_1, \ldots, X_k$ be vector fields spanning a k-dimensional distribution on the Riemannain metric space, (M,ρ). Define

$V_0 = \{0\}$,
$V_1 = Span\{X_i\}, \ldots,$
$V_n = V_{n-1} + Span\{[X,Y], X \in V_1, Y \in V_{n-1}\}$

The X_i are said to satisfy **Hörmander's Condition** at $p \in M$ if $V_j(p) = T_pM$ for some $j \leq d$. Such a set of vector fields is often called "**bracket-**



generating." The X_i are said to satisfy the **strong bracket-generating hypothesis** if for each $Y \in V_1$, $V_1 + Span\{[X,Y], X \in V_1\} = T_pM$.

Theorem 2.1 (Chow) *Suppose $X_1, \ldots, X_k$ satisfy Hörmander's condition at $m \in M$. Then any p sufficiently close to m may be joined to m by a horizontal path.*

It follows from the proof of this theorem that the topologies generated by ρ and ρ_S are the same when the X_i are bracket-generating. However, the metrics themselves are not (Lipschitz) equivalent under these conditions (ie: there do not exist numbers $0 < a < b$ such that $a\rho(p,q) \le \rho_S(p,q) \le b\rho(p,q) \forall p,q \in M$.) One may find a more detailed explanation in **[S]** or **[Gr1]**. Looking at the geometric interpretation of the Lie bracket, it is clear why Hörmander's Condition is sufficient to ensure connectivity.

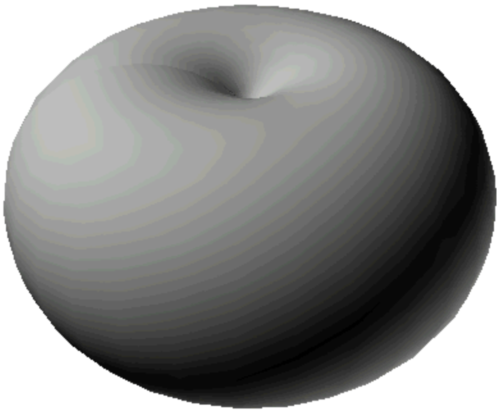
For small $\epsilon > 0$, start at $p \in M$ and draw the integral curve of X for time ϵ. Now draw the integral curve of Y (also for time ϵ), then the integral curve of $-X$, and finally the integral curve of $-Y$. The vector $[X,Y](p)$ points (nearly) in the direction of the endpoint to this path. Thus, if X and Y are admissible tangent vector fields, then (locally) there is a path connecting p to points along the integral curve of $[X,Y](p)$.

Corollary 2.1.1 *If M is connected and $X_1, \ldots, X_k$ satisfy Hörmander's condition at each $m \in M$, then the metric ρ_S is finite.*

For arbitrary $p \in M$, the set of points, S, which can be joined to p by a finite sequence of horizontal curves is both open and closed . The fact that S and $M \backslash S$ are open is immediate from Chow's theorem**[Gr1]**. Since M is connected, this implies S=M.

3 A Contact Structure on $\mathbf{R}^3$

For a Riemannian manifold M of dimension $2n+1$, a **contact 1-form**, α, is one for which $\alpha \wedge d\alpha \wedge ... \wedge d\alpha$ (n factors of $d\alpha$) never vanishes. If α is contact 1-form



and Y its dual vector field (ie: $\alpha(Y) \equiv 1$ and $d\alpha(Y, X) = 0$ for all X), there is a $2n$-dimensional distribution $X_1, \ldots, X_n$ ($Y \notin Span\{X_i\}$) which is strong bracket-generating **[Gr1]**.

Example 2 For $M = \mathbf{R}^3$, let $\alpha =-ydx + xdy + dz$. The dual vector field to α is $\xi = \frac{1}{1+x^2+y^2}(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y} + \frac{\partial}{\partial z})$. The orthogonal complement to ξ is spanned by $X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $X_2 = \frac{\partial}{\partial y}-x\frac{\partial}{\partial z}$. Then $\{X_1, X_2\}$ defines a strong bracket-generating distribution, with $X_3 = [X_1, X_2] = 2\frac{\partial}{\partial z}$ and $\xi \notin Span\{X_1, X_2\}$.

The locus of points with a fixed distance from the origin resembles a pinched sphere which is not smooth at the north and south poles.

Near the $x-y$ plane, distance is changed very little, which makes the surface sphere-like. For points which are on this locus near the z-axis, the distance to the origin has increased greatly, so they appear closer in Euclidean space.

4 Control Systems

The previous example arises in the theory of optimal control (see **[B1]**), where it is useful in solving the following system of differential equations:

$$\dot{x} = u_1$$
$$\dot{y} = u_2$$
$$\dot{z} = u_1 y - u_2 x$$

Here, $u_1(t, p)$ and $u_2(t, p)$ are real-valued functions of time and position, often called impulse functions. Allowing discontinuous impulse functions generates solutions which are merely piecewise smooth. Writing $p(t) = (x(t), y(t), z(t))$

and recalling the definitions of X_1 and X_2, the above equation can be rewritten as

$$\dot{p} = u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + (u_1 - u_2)\frac{\partial}{\partial z} = u_1 X_1 + u_2 X_2$$

Then the set of admissible paths for Example 2 completely describes the solutions to this differential equation. Starting at p_0 and setting

$$u_1(t,p) = \begin{cases} 1 & \text{for} t \in [0,\epsilon) \text{ and all } p \\ 0 & \text{for } t \in [\epsilon, 2\epsilon) \\ -1 & \text{for } t \in [2\epsilon, 3\epsilon) \\ 0 & \text{for } t \in [3\epsilon, 4\epsilon] \end{cases} \quad \text{and } u_2(t,p) = \begin{cases} 0 & \text{for } t \in [0,\epsilon) \text{ and all } p \\ 1 & \text{for } t \in [\epsilon, 2\epsilon) \\ 0 & \text{for } t \in [2\epsilon, 3\epsilon) \\ -1 & \text{for } t \in [3\epsilon, 4\epsilon] \end{cases}$$

the endpoint to the path at $t = 4\epsilon$ is nearly in the direction $[X_1, X_2](p_0)$, as in the figure for Section 2.

The fact that $\{X_1, X_2\}$ is a bracket-generating distribution implies that any two points may be connected along a solution to the differential equation. This is of interest to the control theorist since it implies that in a system governed by the above equations, all states are reachable from any initial one. The geodesics for this singular Riemannian structure satisfy

$$\begin{gathered} \ddot{x} + \lambda \dot{y} = 0 \\ \ddot{x} - \lambda \dot{x} = 0 \\ \ddot{z} + \lambda(x\dot{x} + y\dot{y}) = 0 \end{gathered}$$

for a suitable constant λ. Rewritten, the equation reads

$$\ddot{p} = \lambda(-u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial x} + (u_1 x + u_2 y)\frac{\partial}{\partial z}) = \lambda(u_1 X_2 - u_2 X_1)$$

Recall that $\dot{p} = u_1 X_1 + u_2 X_2$. At each point, $\ddot{p}$ is orthogonal to $\dot{p}$, as is expected of geodesics. More importantly, both these vector fields lie in the distribution spanned by X_1 and X_2. By solving this system, one finds that every point along the z-axis is conjugate to the origin. This is equivalent to saying that each geodesic emanating from the origin passes through every point on this line without ever traveling along it. Since the distance between conjugate points is bounded below by $\frac{\pi}{\sqrt{K}}$, it follows that the sectional curvature (taken with respect to the singular Riemannian metric) is unbounded in every neighborhood of the origin. Still more surprising is the fact that $\mathbf{R}^3$ has Hausdorff dimension 4 with respect to this metric.

5 Hausdorff Dimension

Given a Riemannian metric space (M,ρ) and an arbitrary metric, σ, one defines the Hausdorff dimension of (M,σ) at $m \in M$ to be $D = D(m)$ if there exist positive constants c,C and ϵ such that:

$$c < \frac{Vol_\rho B_r^\sigma(m)}{r^D} < C \forall r < \epsilon,$$

where Vol_ρ is the Riemannian volume element and $B_r^\sigma(m) = \{x \in M : \sigma(m,x) < r\}$. This must be defined locally, since a uniform value does not generally exist. The (global) **Hausdorff dimension** of M is then defined as $dim_H(M,\sigma) = \sup_{m \in M} D(m)$.

When σ is a Riemannian metric, $Vol_\rho B_r^\sigma(m) \approx r^D$ (for small $r > 0$) at all points of the manifold. Hence, $dim_H = d = dim_{top}$ in this case. A singular Riemannian metric, however, can display quite different behavior.

Suppose $\{X_1, \ldots, X_k\}$ spans a distribution on M, and that the vector spaces V_i have been definined accordingly. If X is a vector field on M, one defines $degX(m) = n$ when $X(m) \in V_n$ and $X(m) \in V_{n-1}(m)$, with the assumption that $degX = 0$ if $X \notin V_n$ for any n. The distribution spanned by $\{X_i\}$ is said to be **generic** when $degX_i$ is the same at all points of the manifold, and the X_i are said to be **equiregular** in this case.

Theorem 5.1 (Mitchell) *If $X_1, \ldots, X_k$ span a generic distribution on M, then the Hausdorff dimension of the space (M, ρ_S) is given by*

$$\sum_{i=1}^{n} i(dimV_i - dimV_{i-1})$$

where $dimV_i$ is the linear dimension of the vector space V_i. This gives us the expected dimensions of 2 and 4 in the previous examples.

In Example 2, the singular Riemannian metric, ρ_S, is equivalent to ρ along integral curves of X_1 or X_2. However, along an integral curve of X_3, ρ_S is locally equivalent to $\sqrt{\rho}$, so that if for example $x_0 = (0,0,0)$ and $x_r = (0,0,r)$,

$$c_1\sqrt{r} \le \rho_S(x_0, x_r) \le c_2\sqrt{r} \text{ for } 0 < r < \epsilon.$$

Since ρ is locally equivalent to ρ_S^2 along X_3 and equivalent to ρ along X_1 and X_2,

$$C_1(r)(r)(r^2) \le Vol_\rho B_r^{\rho_S} \le C_2(r)(r)(r^2)$$

for small $r > 0$. Arguments are similar in other cases, with ρ_S locally equivalent to $\sqrt[n]{\rho}$ along an integral curve of X whenever $degX = n$.

Suppose that $\{X_1, \ldots, X_k\}$ induces a singular Riemannian structure on (M,ρ) and that $\{X_1, \ldots, X_k, \ldots, X_d\}$ is an equiregular basis of $T_p m$ at each $p \in M$. Let n_j be the order of the set $\{X_i : degX_i = j\}$. Then the dimension of V_i is $n_1 + \cdots + n_i$, so that $dimV_i$–$dimV_{i-1} = n_i$. Then $dim_H(M, \rho_S) = \sum_{i=1}^{d} degX_i$. As anticipated, this formula implies that the Hausdorff dimension is the sum of the degrees of the X_i. Further details may be found in **[M]**.

Although the contact structure on $\mathbf{R}^3$ does not change its local topology, the image of this manifold under any isometric imbedding into Euclidean space has Hausdorff dimension four. One may think of such a Riemannian manifold as having been stretched and folded at all scales, making it the higher-dimensional counterpart to a space-filling curve. In cases such as the first example, where $dim_H < dim_{top}$ there is no imbedding of (M,ρ_S) into Euclidean space. The space is no longer paracompact in this setting. In cases with $dim_H \geq dim_{top}$, such an imbedding may exist, and the Hausdorff dimension measures the complexity of its image.

6 Completeness

As in Riemannian geometry, one says that (M,ρ_S) is **complete** if it is complete as a metric space. The following counterpart to the Hopf-Rinow theorem relates this to geodesic completeness in the singular Riemannian case.

Theorem 6.1 *Suppose that M is connected.*

(a) If M is complete, then every geodesic can be extended indefinitely, and any two points can be joined by a geodesic.

(b) Assume the strong bracket-generating hypothesis. If there exists a point P such that every geodesic from P can be indefinitely extended, then M is complete.

A Riemannian metric, ρ, is said to be a **Riemannian contraction** of ρ_S if there is a singular Riemannian structure on (M,ρ) which induces the metric ρ_S. Such a contraction is not unique, and as suggested by the wording in the following theorem, it is possible that a Riemannian contraction of a complete singular Riemannian metric may fail to be complete.

Theorem 6.2 *Let M be a singular Riemannian manifold. If there exists a Riemannian contraction of the metric with respect to which M is complete, then M is complete in the given singular Riemannian metric.*

Proofs of these theorems may be found in **[S]**.

7 Connection to Nilpotent Lie Groups

Let $G = G_0$ be a group. Recursively define $G_n = \{aba^{-1}b^{-1} : a, b \in G\}$. Then G is said to be **nilpotent** if $G_k = \{e\}$ for some k. The G_n are defined in a similar fashion to the V_n (see Section 2.) It is for this reason and because of the relationship between the commutators of a Lie group and the bracket of elements in its Lie algebra that a nilpotent Lie group admits a bracket-generating distribution whenever $G_{k-1} \neq \{e\}$ and $G_k = \{e\}$ for $k > 1$.

Example 3 The Heisenberg group, G, of matrices is a nilpotent non-Abelian Lie group which is diffeomorphic to $\mathbf{R}^3$. There is a basis $\{X_i\}$ of $\mathfrak{g}$ with $[X_1, X_2] = X_3$ and $[X_1, X_3] = [X_2, X_3] = 0$. Let ρ be the Riemannian metric for which these vector fields are orthonormal. Then ρ is represented by the matrix I_3 with respect to $\{X_i\}$.

One can define a family $\{\phi_t\}$ of automorphisms of $\mathfrak{g}$, where ϕ_t is represented with respect to $\{X_i\}$ by the matrix

$$A_t = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}$$

Each of the ϕ_t defines a new Riemannian inner product on $\mathfrak{g}$, with $\langle X_i, X_j \rangle_t = A_t(i,j)$, the ij-th entry of A_t. Let σ_t be the Riemannian metric on G induced by this inner product. The ϕ_t exponentiate to automorphisms Φ_t of G, and Φ_t maps (G, ρ) isometrically to (G, σ_t) . Then $(G, \rho/t)$ is isometric to $(G, \sigma_t/t) = (G, \phi_t^*/t)$.

Now, $\phi_t^*(\rho)/t$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}$$

As $t \to \infty$, lengths of the vectors X_1 and X_2 remain unchanged, while the length of X_3 becomes infinite. Taking the limit of these Riemannian metrics, one obtains a singular Riemannian metric, ρ_S, induced by the distribution spanned by X_1 and X_2. Suppose p and q lie along an integral curve, γ , of X_3. As $t \to \infty$, the length of this curve becomes very large. At some point, there will be a path whose tangent vector field lies in H and whose length is less than that of γ. In the limit, all geodesics have will tangent vector fields which lie in H. Then the sequence of metric spaces (G, σ_t) converges to the singular Riemannian metric space (G, ρ_S), where convergence is defined in terms of the Gromov-Hausdorff limit of metric spaces (see **[Wh]** for its definition.)

At first it seems odd that σ_t/t converges to a finite metric rather than to zero. The confusion lies in the use of the word "metric." It is understood to mean the distance function when applied to two points of the manifold and to mean the inner product when applied to two elements of the Lie algebra. Here σ_t/t represents an inner product on T_pM, with $\langle X_1, X_1\rangle_t = \langle X_2, X_2\rangle_t = 1$, $\langle X_3, X_3\rangle_t = t$, and $\langle X_i, X_j\rangle_t = 0$ otherwise. The length of paths along X_1 and X_2 is unchanged and those along X_3 are multiplied by a factor of t. However, the distance between two points, p and q, along an integral curve of X_3 is not merely scaled by t. Their distance remains bounded and converges to $\rho_S(p,q)$ as t tends to ∞. The distance between points along integral curves of X_1 or X_2 remains the same.

Theorem 7.1 (Pansu) *If G is a nilpotent Lie group with left-invariant Riemannian metric ρ, then $\lim_{t\to\infty}(G, \rho/t) = (G', \rho_S)$, where G' is the graded Lie group associated to G and ρ_S a singular Riemannian metric on G'.*

Now consider $\lim_{t\to 0}(M, \rho/t) = \lim_{t\to\infty}(M, t\rho)$ for an arbitrary manifold M with Riemannian metric ρ. One defines the **tangent cone of** M based at $m \in M$ to be $T_mM = \lim_{t\to\infty}(M, t\rho)$ whenever this limit exists.

Theorem 7.2 (Mitchell) *For a generic distribution, H, on M, the tangent cone at $m \in M$ is isometric to (G, ρ_S), where G is a nilpotent Lie group and ρ_S a singular Riemannian metric on G.*

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