$Spin^c$-MANIFOLDS

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1. Introduction

$Spin^c$-structures on manifolds are a complex analogue to the more common notion of spin structures on manifolds. They have been known since the 1960's (see [**A-B-S**]), but they had no real importance (as far as I can tell), until the recent announcement of the Seiberg-Witten equations for 4-manifolds in [**W**]. These equations promise to vastly simplify the study of smooth 4-manifolds, and their definition requires the presence of a $spin^c$-structure. In this paper I will review the definition of $spin^c$-structures on manifolds from both a geometric and algebraic point of view, and prove their existence in some important cases. I will conclude by looking at how they appear in the formulation of the Seiberg-Witten equations.

2. Geometric formulation of $Spin_n^c$

In one sense, *spin* and $spin^c$ structures are just generalizations of orientations. Consider a smooth manifold M^n with tangent bundle TM. This vector space bundle gives rise to a principal $O(n)$-bundle of frames, which we denote $P_O(TM)$. Recall that the manifold is said to be *orientable* if this bundle can be reduced to an $SO(n)$-bundle $P_{SO}(TM)$, making the fibers connected. This means that any trivialization of the bundle over the (disconnected) 0-skeleton of M can be extended to a trivialization over the (connected) 1-skeleton. The next step is to make the fiber simply connected (where possible). This will mean that a trivialization over the 1-skeleton of M can be extended over the 2-skeleton. Recalling that, for $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}_2$, we define $Spin_n$ to be the double cover of $SO(n)$. For $n \geq 3$, this is the universal (i.e. simply- connected) cover; in the exceptional cases we have $Spin_2 = S^1$ and $Spin_1 = S^0$. We then say that the manifold is *spin* if the bundle $P_{SO}(TM)$ has a double cover by a principal $Spin_n$-bundle $P_{Spin}(TM)$.

To find the complex analogue, we replace $SO(n)$ by the group $SO(n) \times U(1)$, and consider its double cover. With this in mind, we define:

$$Spin_n^c = (Spin_n \times U(1))/\{\pm(1,1)\} = Spin_n \times_{\mathbb{Z}_2} U(1)$$

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This is the desired double cover of $SO(n) \times U(1)$ via the map $[A, \lambda] \mapsto [p(A), \lambda^2]$, where p is the double cover of $SO(n)$ by $Spin_n$. Finally, we define M to be *spin*^{c} if given the bundle $P_{SO}(TM)$, there are principal bundles $P_{U(1)}(TM)$ and $P_{Spin^c}(TM)$ with a *spin*^{c}-equivariant bundle map:

$$\xi : P_{Spin^c}(TM) \longrightarrow P_{SO}(TM) \times P_{U(1)}(TM).$$

This definition of $Spin_n^c$ leads to a very nice geometric criterion for the existence of a *spin*^{c}-structure (**[K2]**). Since $U(1) = SO(2)$, there is a natural map $SO(n) \times U(1) \to SO(n+2)$ which extends (via Whitney sum) to a map of bundles. We can define $Spin_n^c$ as the pullback by this map of the covering map $Spin_{n+2} \to SO(n+2)$:

$$\begin{array}{ccc} Spin_n^c & \longrightarrow & Spin_{n+2} \\ \downarrow & & \downarrow \\ SO(n) \times U(1) & \longrightarrow & SO(n+2) \end{array}$$

Therefore, a *spin*^{c}-structure on TM consists of a complex line bundle L and a *spin*-structure on $TM \oplus L$. We can restate this as:

Theorem 1. *A manifold M is spin^c (i.e. TM has a spin^c-structure) $\Leftrightarrow$ there is a complex line bundle L over M such that $TM \oplus L$ has a spin-structure.*

So M is *spin*^{c} if the obstruction to extending a trivialization of the tangent bundle over the 2-skeleton can be removed by adding a complex line bundle.

3. Examples of $Spin^c$-manifolds

We start with examples of manifolds which have canonical $Spin^c$-structures.

Theorem 2. *If M is a spin manifold, then M has a canonical spin^c-structure.*

Proof: We simply extend the spin structure by taking the fiber product with the trivial $U(1)$-bundle U_1, letting

$$P_{Spin^c}(TM) = P_{Spin}(TM) \times_{M,\mathbb{Z}_2} U_1. \square$$

Theorem 3. *If M has an almost complex structure, then M has a canonical spin^c-structure.*

Proof: Let $j : U(k) \to SO(2k)$ denote the natural homomorphism. Then we can define a homomorphism $g : U(k) \to SO(2k) \times U(1)$ by $g(A) = (j(A), det(A))$. Although j does not lift to $Spin_{2k}$, g *does* lift to $Spin_{2k}^c$. Denote this lift γ. An almost complex structure on M means TM can be viewed as a complex vector bundle, and so M has an unitary frame bundle $P_{U(n)}(TM)$. We now construct the desired $Spin^c$ bundle as an associated bundle:

$$P_{Spin^c}(TM) = P_{U(n)}(TM) \times_\gamma Spin^c_{2k}. \ \square$$

In fact, we can give another, more algebraic, general criterion for whether a manifold has a $Spin^c$-structure:

Theorem 4. *An orientable manifold M can be given a $Spin^c$-structure $\Leftrightarrow$ the second Stiefel-Whitney class $w_2(M)$ is the mod 2 reduction of an integral class.*

PROOF: Recall that a manifold M has a *spin*-structure $\Leftrightarrow$ the second Stiefel-Whitney class $w_2(M)$ is 0 (see **[L-M]** and **[K2]**). So we apply our geometric criterion from the last section, which says that M can be given a $spin^c$-structure $\Leftrightarrow$ there is a complex line bundle L such that $TM \oplus L$ is spin, which means $w_2(TM \oplus L)$ is 0. But, since the Stiefel-Whitney classes are stable, we have:

$$w_2(TM \oplus L) = w_2(TM) + w_2(L) + w_1(TM)w_1(L) = 0$$

Both these bundles are orientable, so the first Stiefel-Whitney classes are both 0, which means $w_2(TM) + w_2(L) = 0$. Since these are mod 2 classes, $w_2(TM) = w_2(L)$. $w_2(L)$ has an integral lift, the first Chern class of the line bundle, so $w_2(TM) = w_2(M)$ also has an integral lift, which proves the theorem in one direction. To go the other way, we can follow the same argument backwards, since if $w_2(TM)$ lifts to an integral class e, we can always find a complex line bundle with first Chern class e, which will be the line bundle we need for our $spin^c$-structure. $\square$

In particular, by **[M]**, this means that any orientable four manifold can be given a $Spin^c$-structure, which will be crucial to the formulation of the Seiberg-Witten equations.

4. CLASSIFICATION OF $spin^c$-STRUCTURES OF A MANIFOLD

We will classify $spin^c$-structures by using classifying spaces, an important tool from algebraic topology. Our discussion here follows **[?]**. We start with a basic definition:

DEFINITION: A *classifying space* for a group G is a CW-complex BG and principal G-bundle EG over BG such that given any space X and a principal G-bundle E over X, there is a map $f : X \to BG$ such that $E = f^*(EG)$.

It is not hard to show that BG is unique up to homotopy equivalence. From our definition and discussion of $Spin_n^c$ we have the following commutative diagram of groups, with rows and columns exact:

$$\begin{array}{ccccc} \mathbb{Z}_2 & \subset & U(1) & \rightarrow & U(1) \\ \| & & \downarrow & & \downarrow \\ \mathbb{Z}_2 & \subset & Spin_n^c & \rightarrow & SO(n) \times U(1) \\ & & \downarrow & & \downarrow \\ & & SO(n) & = & SO(n) \end{array}$$

This diagram induces a similar commutative diagram of classifying spaces (by, for example, Milgram's construction of the classifying space in **[P]**). Therefore, we can view $BSpin_n^c$ as a bundle over $BSO(n)$ with fiber $BU(1)$.

Now we view the tangent bundle of a manifold M as a map $\eta : M \rightarrow BSO(n)$. A *spin*^{c}-structure on the tangent bundle is then a lift of this map to $BSpin_n^c$, giving a commutative diagram:

$$\begin{array}{ccc} BU(1) & \rightarrow & BSpin_n^c \\ & \nearrow & \downarrow \\ M & \xrightarrow{\eta} & BSO(n) \end{array}$$

Theorem 5. *The set of lifts of η is in bijective correspondence with [M,BU(1)].*

Proof: Let h_p denote the homeomorphism from $BU(1)$ to the fiber of $BSpin_n^c$ over the point $p \in BSO(n)$. Given a map $\lambda \in [M, BU(1)]$, define the lift η_λ by $\eta_\lambda(x) = h_{\eta(x)} \circ \lambda(x)$. This is clearly an injective map from $[M, BU(1)]$ into the set of lifts; it is also surjective, since two different lifts will have to disagree on at least one fiber. □

Since $[M, BU(1)]$ is just the set of complex line bundles over M, which are classified by their first Chern class, the theorem implies that the set of lifts (and hence the *spin*^{c}-structures on M) is in correspondence with the second cohomology group $H^2(M; \mathbb{Z})$. (Alternatively, we note from **[P]** that $BU(1) = BS^1 = \mathbb{CP}^\infty$. Since $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, the Eilenberg-Maclane space, this means $[M, BU(1)] = [M, K(\mathbb{Z}, 2)] = H^2(M; \mathbb{Z})$, by **[K1]**.) We can combine this group structure with the correspondence to define a simply transitive group action of $[M, BU(1)] = H^2(M; \mathbb{Z})$ on the set of lifts:

$$\gamma \cdot \eta_\lambda = \eta_{\gamma \cdot \lambda}$$
$$\gamma, \lambda \in H^2(M; \mathbb{Z})$$

We also want to consider our geometric criterion identifying a *spin*^{c}-structure on M with a complex line bundle L over M and a *spin*-structure on $TM \oplus L$. The first question is whether the *spin*^{c}-structure determines the complex line bundle in this description. The answer is "Yes." From the commutative diagram of groups drawn above, we can induce the following commutative diagram:

$$\begin{tikzcd} & BSpin_n^c \arrow[d] \arrow[dr, "pr"] & \\ M \arrow[ur, "\mu"] \arrow[dr, "\eta"'] & & B(SO(n)\times U(1)) \arrow[dl] \\ & BSO(n) & \end{tikzcd}$$

where the map $\mu : M \to BSpin_n^c$ is a lift of the map $\eta : M \to BSO(n)$, and the maps on the right-hand side of the diagram are projections induced from our commutative diagram of groups. So the lift μ of η canonically gives us a lift $pr \circ \mu : M \to B(SO(n) \times U(1))$. This lift is the complex line bundle desired.

We can also ask the question in reverse: does the complex line bundle determine the $spin^c$-structure? Here, the answer is unsurprisingly "No." Recall from the proof of Theorem 4 in Section 3 that we must have $w_2(TM) = w_2(L) = c_1(L) \bmod 2$. Hence there are strictly less than $|H^2(M;\mathbb{Z})|$ possible line bundles, so these cannot determine the $|H^2(M;\mathbb{Z})|$ $spin^c$-structures in a one-to-one fashion. The question now becomes: given a complex line bundle, how many different $spin^c$ -structures are associated with that bundle?

As a first approximation, we compute the number of $spin$-structures on $TM \oplus L$. As above, the $spin$-structures on $TM \oplus L$ correspond to lifts of a map $\eta : M \to BSO(n+2)$ to $BSpin_{n+2}$, so we have a diagram:

$$\begin{tikzcd} B\mathbb{Z}_2 \arrow[r] & BSpin_{n+2} \arrow[d] \\ M \arrow[ur] \arrow[r, "\eta"] & BSO(n+2) \end{tikzcd}$$

Exactly as in the previous theorem, we find that the set of lifts is in bijective correspondence with $[M, B\mathbb{Z}_2]$. **[P]** proves that $B\mathbb{Z}_2 = \mathbb{RP}^\infty$. But $\mathbb{RP}^\infty$ is just the Eilenberg-Maclane space $K(\mathbb{Z}_2, 1)$, so we have $[M, B\mathbb{Z}_2] = [M, K(\mathbb{Z}_2, 1)] = H^1(M;\mathbb{Z}_2)$ (the last equality is proved in **[K1]**). Therefore, the set of $spin$-structures on $TM \oplus L$ corresponds to $H^1(M;\mathbb{Z}_2)$.

While each of these $spin$-structures pulls back to a different lift from $B(SO(n)\times U(1))$ to $BSpin_n^c$, they are not all different when considered as lifts from $BSO(n)$ to $BSpin_n^c$. We will not completely answer the question of when they are or are not different, but we will show:

Theorem 6. *Two lifts which differ by the action of an element in $H^1(M;\mathbb{Z}_2)$ which comes from $H^1(M;\mathbb{Z})$ give the same $spin^c$-structure, assuming the complex line bundles are the same.*

PROOF: As above, we have that $H^1(M;\mathbb{Z}) = [M, K(\mathbb{Z},1)] = [M, S^1]$. It will clearly suffice to show that a lift corresponding to an element in $H^1(M;\mathbb{Z}_2)$ which comes from $H^1(M;\mathbb{Z})$ gives the same $spin^c$- structure as the lift corresponding to the 0 element. Such a lift would factor through S^1 in each fiber; i.e. the image of the lift in each fiber $B\mathbb{Z}_2 = \mathbb{RP}^\infty$ lies in the canonical copy of S^1 embedded in $\mathbb{RP}^\infty$ as $\mathbb{RP}^1$. However, when we view $BSpin_n^c$ as a bundle over $BSO(n)$, the fiber is $BU(1) = \mathbb{CP}^\infty$, which is simply-connected. Therefore the copies of S^1 can all be homotoped to a point in these fibers (simultaneously, since the homotopy is the same in each fiber), which means the lift is the same as the 0-lift. $\square$

Hence, the number of $spin^c$-structures on M associated with each complex line bundle over M is at most

$$|H^1(M;\mathbb{Z}_2) modulothoseelementscomingfrom H^1(M;\mathbb{Z})|.$$

5. A Description of $Spin_n^c$ via Clifford modules

In this section I will give a much more algebraic formulation of the groups $Spin_n$ and $Spin_n^c$. This formulation will give us information about the structure of these groups which is very useful in studying vector bundles. However, before diving into a sea of algebra, I will try to give some geometrical motivation, following **[K2]**.

Recall that an element of the orthogonal group $O(n)$ can always be written as a product of reflections ρ_i across hyperplanes through the origin. Each such reflection is determined by a unit normal v_i to the hyperplane; note that v_i and $-v_i$ determine the same reflection. So we can write an element of $O(n)$ as a "product" $[v_1 \cdot v_2 \cdot \cdots \cdot v_k]$, where each equivalence class contains a product and its negative, and $0 \leq k \leq n$. Then the double cover of $O(n)$ is just the group of signed products, which is called Pin_n (a play on $SO(n)$ and $Spin_n$ which stuck). We will define the *Clifford algebra* $C\ell_n$ so that it contains Pin_n in a natural way.

DEFINITION: Given a real vector space V with an inner product Q, the *Clifford algebra* $C\ell(V,Q)$ is the quotient algebra $\mathcal{T}(\mathcal{V})/\mathcal{I}(\mathcal{V})$, where $\mathcal{T}(\mathcal{V})$ is the tensor algebra $\bigotimes V$, and $\mathcal{I}(\mathcal{V})$ is the ideal generated by elements of the form $v \otimes v - Q(v,v)$.

To increase the resemblance to our geometric motivation (and to make things easier to write) we will usually write products as vw rather than $v \otimes w$. The relation given in the definition can be rewritten as $vw + wv = 2Q(v,w)$. These relations have a particularly nice form when we consider an orthonormal basis $\{e_1, \ldots, e_n\}$ for V, and assume that Q

is positive definite. Then we have that $e_ie_j = -e_je_i$ and $e_ie_i = 1$. From these, we can see that a basis for $C\ell(V,Q)$ is $\{e_I = e_{i_1}\dots e_{i_k}$ where $i_1 < i_2 < \cdots < i_k,$ and $0 \le k \le n\}$ (when $k = 0$ we get the identity $1 = e_{\emptyset}$). Therefore, the dimension of $C\ell(V,Q)$ is 2^n, where n is the dimension of V.

$C\ell(V,Q)$ has a natural $\mathbb{Z}_2$-grading $C\ell(V,Q) = C\ell^0(V,Q) \oplus C\ell^1(V,Q)$ where the first term is generated by products of an even number of elements of V, and the second is generated by products of an odd number of elements of V. We consider the multiplicative group of units in the Clifford algebra, denoted $C\ell^{\times}(V,Q)$. This group has a natural representation in the Clifford algebra, called the *adjoint* representation:

$$Ad : C\ell^{\times}(V,Q) \longrightarrow Aut(C\ell(V,Q))$$

$$Ad(\varphi)(x) = \varphi x \varphi^{-1}$$

If $v \in V$ with $Q(v,v) \neq 0$, then v is a unit ($v^{-1} = -v/Q(v,v)$), and $Ad(v)$ preserves the inner product ($Q(Ad(v)(w), Ad(v)(w)) = Q(w,w)$); so Ad restricts to a representation of $P(V,Q) = \{v \in V$ s.t. $Q(v,v) \neq 0\}$ in $O(V,Q) = \{\lambda \in GL(V)$ preserving $Q\}$. Now we define:

$$Pin(V,Q) \subset P(V,Q) \text{ is the subgroup generated by } v \in V \text{ with } Q(v,v) = \pm 1$$

$$Spin(V,Q) = Pin(V,Q) \cap C\ell^0(V,Q)$$

We can show that these groups (for a real vector space) are double covers of $O(V,Q)$ and $SO(V,Q)$ respectively, so this agrees with our geometric definition of the spin groups.

We are particularly interested in the case when $V = \mathbb{R}^n$, and Q is the usual positive definite inner product (dot product). Then we define $C\ell_n = C\ell(V,Q)$, $Spin_n = Spin(V,Q)$, etc. We now define the groups $Spin_n^c$ as before:

$$Spin_n^c = Spin_n \times_{\mathbb{Z}_2} U(1)$$

We associate with $C\ell_n$ a *volume element* $\omega = e_1e_2\cdots e_n$, where $\{e_1,\dots,e_n\}$ is an orthonormal basis for $\mathbb{R}^n$ (with a given orientation). ω is independent of the choice of this basis (in $C\ell_n$), and we have the relation:

$$\omega^2 = (-1)^{n(n+1)/2}$$

Similarly, we consider the case when V is a complex vector space and define $\mathbb{C}\ell_n$ to be $C\ell_n \otimes \mathbb{C}$. Notice that $Spin_n^c \subset \mathbb{C}\ell_n$. Again, we define a volume element $\omega_{\mathbb{C}} = \imath^{[(n+1)/2]}\omega$. In this case, we find the square of the volume element is always 1.

These volume elements give us useful decompositions of vector spaces which have $C\ell_n$- representations.

Definition: A $C\ell_n-module$ is a real vector space W together with a representation $\rho : C\ell_n \to Hom_{\mathbb{R}}(W,W)$. We often denote $\rho(\varphi)(w)$ by

$\varphi \cdot w$, and call this operation *Clifford multiplication.* Similarly, in the complex case we define $\mathbb{C}\ell_n$-modules.

If W is a $C\ell_n$-module, and $\omega^2 = 1$, then we get a decomposition $W = W^+ \oplus W^-$ into the eigenspaces of ω, so $W^\pm = (1/2)(1 \pm \omega)W$. In the complex case, the square of the volume element is always 1, so the decomposition always exists.

We say that the representation ρ is *reducible* if W can be written $W_1 \oplus W_2$, where $\rho(\varphi)(W_i) \subseteq W_i$ for every $\varphi \in C\ell_n$. Otherwise, we call the representation *irreducible.* We call two representations $\rho_j : C\ell_n \to Hom(W_j, W_j)$ *equivalent* if there is a linear isomorphism $F : W_1 \to W_2$ such that $F \circ \rho_1(\varphi) \circ F^{-1} = \rho_2(\varphi)$ for every $\varphi \in C\ell_n$. There is a well-understood classification of Clifford algebras (see **[L-M]**) which gives us the following fact:

Theorem 7. *The number of inequivalent irreducible real representations of $C\ell_n$ is 2 if $n+1 \equiv 0 \pmod 4$, and 1 otherwise. The number of inequivalent irreducible complex representations of $\mathbb{C}\ell_n$ is 2 if n is odd and 1 if n is even.*

Finally, we will introduce one more type of bundle - the *spinor bundles* of a manifold:

Definition: If the manifold M has a spin structure $\xi : P_{Spin}(TM) \to P_{SO}(TM)$, a *real spinor bundle* is an associated bundle $S(M) = P_{Spin}(TM) \times_\mu W$, where W is a left module for $C\ell_n$ and $\mu : Spin_n \to SO(W)$ is the representation given by Clifford multiplication by elements of $Spin_n \subset C\ell_n^0$. Similarly, we define a *complex spinor bundle*, with W a complex left module for $\mathbb{C}\ell_n = C\ell_n \otimes \mathbb{C}$.

We easily generalize this definition to $spin_c$-manifolds by defining the spinor bundle $S(M) = P_{Spin^c}(TM) \times_\triangle V$, where V is a complex $C\ell_n$-module, and $\triangle : Spin_n^c \to GL(V)$ is the restriction of the $C\ell_n$ representation to $Spin_n^c \subset C\ell_n \otimes \mathbb{C}$. If this representation is irreducible, we say that the spinor bundle is *fundamental.* So by the theorem above, there is one fundamental spinor bundle if n is even, and two if n is odd. However, in the odd case the two bundles are equivalent when restricted to $Spin_n^c$, so in fact there is always a unique fundamental spinor bundle, which we denote $S(M)$. Since we are in the complex case, we can use the volume element $\omega_\mathbb{C}$ to decompose $S(M)$ into two bundles $S^\pm(M) = (1/2)(1 \pm \omega_\mathbb{C})S(M)$. We will use these bundles in the next section to define the Seiberg-Witten equations.

6. The Seiberg-Witten equations

To define the Seiberg-Witten equations, we specialize to the case of orientable 4-manifolds, following **[T]** and **[A]**. We know, from section 3, that any orientable 4-manifold has a $spin^c$-structure. We also know, from the classification of Clifford algebras in **[L-M]**, that $\mathbb{C}\ell_4 = \mathbb{C}(4)$, the algebra of 4×4 complex matrices. The unique irreducible complex representation is the natural representation of this group on $\mathbb{C}^4$, so the fundamental spinor bundle $S(M)$ is a $\mathbb{C}^4$-bundle, which splits (as described in section 4) into two $\mathbb{C}^2$-bundles $S^{\pm}(M)$. By restricting this representation to the natural copy of $\mathbb{R}^4$ lying inside $\mathbb{C}\ell_4$, Clifford multiplication gives us a map c from the cotangent bundle $T^*(M)$ into the skew-adjoint endomorphisms of $S(M) = S^+(M) \oplus S^-(M)$ (skew-adjoint because of the relation $vv = -Q(v,v)$). c induces the following map by duality:

$$\sigma : S^+(M) \otimes T^*(M) \to S^-$$

$$\sigma(s \otimes v) = p_-(c(v)(s,0))$$

where p_- is the projection $S(M) \to S^-(M)$.

We will construct the fundamental spinor bundles $S^{\pm}(M)$ explicitly as associated bundles to representations. First, we recall the following Lie group isomorphisms:

$$Spin_4 = SU(2) \times SU(2)$$

$$SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$$

$$Spin_4^c = (SU(2) \times SU(2) \times U(1))/\{\pm 1\}$$

These give us two natural actions of $SO(4)$ on $\mathbb{R}^3$:

$$\lambda_{\pm} : SO(4) \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\lambda_+ : ([p,q], x) \longmapsto Im(px)$$

$$\lambda_- : ([p,q], x) \longmapsto Im(qx)$$

where we are identifying $SU(2) = S^3$ with the unit quaternions, and $\mathbb{R}^3$ with the imaginary quaternions. The associated $\mathbb{R}^3$-bundles to these representations are isomorphic to Λ_+ (the self-dual two-forms) and Λ_- (the anti-self-dual two-forms) respectively. We extend these actions to actions of $Spin_4^c$ on the quaternions $\mathbb{H}$:

$$s_{\pm} : Spin_4^c \times \mathbb{H} \longrightarrow \mathbb{H}$$

$$s_+ : ([p,q,\lambda], x) \longmapsto px\lambda^{-1}$$

$$s_- : ([p,q,\lambda], x) \longmapsto qx\lambda^{-1}$$

We view the associated $\mathbb{R}^4$-bundles to these actions as $\mathbb{C}^2$-bundles, and by **[A]** these are the spinor bundles $S^+(M)$ and $S^-(M)$, respectively. Then we have a pairing:

$$(,) : S^+(M) \otimes S^+(M)^* \longrightarrow \Lambda_+$$

which is the equivariant extension of the map on fibers given by:

$$(,) : x \otimes y \longmapsto Im(xiy)$$

where the bundle of imaginary quaternions is identified with Λ_+ as before.

Our penultimate step is to introduce the complex line bundle $L = det(S^+(M))$, together with a connection A. Together with the riemannian connection on $T^*(M)$, A induces a covariant derivative ∇_A on $S^+(M)$ which maps sections of $S^+(M)$ to sections of $S^+(M) \otimes T^*(M)$. We define the Dirac operator D_A as the composition of this map with σ:

$$D_A : \Gamma(S^+(M)) \to \Gamma(S^-(M))$$
$$D_A(s)(m) = \sigma(\nabla_A(s)(m))$$

We are now ready to state the Seiberg-Witten equations. The data for these equations is a pair (A, ψ) where A is a connection on L and ψ is a section of $S^+(M)$, and we let F_A^+ denote the self-dual part of the curvature of A:

$$D_A(\psi) = 0$$
$$F_A^+ = (\psi, \psi^*)$$

The Seiberg-Witten invariant is given by properly counting the solutions to these equations, as described in **[T]**. Taubes also states the fundamental theorem:

Theorem 8. *If M is a compact, oriented, connected 4-manifold with $b_2^+ > 1$, then the Seiberg-Witten invariant SW is a map from the space of spin^c-structures on M to the integers $\mathbb{Z}$ which depends only on the underlying smooth structure of M.*

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