Complete Embedded Minimal Surfaces in $\mathbb{R}^3$

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1. Introduction

The study of minimal surfaces was at first naturally motivated by the study of soap films spanning a wire frame. The existence of such a surface for any smooth or piecewise smooth wire frame has been established. But as is mathematicians' wont, the study has long since evolved beyond the objects found in the natural world. One aspect of minimal surfaces that has seen much recent development is the study of complete embedded minimal surfaces. Because of the minimal surface equations, such surfaces in $\mathbb{R}^3$ are necessarily noncompact, thus quite unlikely to appear as soap films or otherwise, yet their study is nonetheless mathematically illuminating, as well as intrinsically beautiful.

One current goal in this field is the classification of complete embedded minimal surfaces. In this paper I will introduce the formal definition of minimal surfaces and the Enneper-Weierstrass method for constructing complete immersed minimal surfaces, and then discuss some of the more recent discoveries and theorems leading towards the goal of classification.

2. Definition of a minimal surface

We begin by defining a minimal surface in $\mathbb{R}^n$ parametrized by a domain $D \subset \mathbb{R}^2$. Specifically, we write $S = x(u, v)$ where $x(u, v)$ is an immersion. Then, for a fixed point $p \in S$, for a given unit tangent vector $t \in T_pS$, a given normal vector $n \in T_p^{\perp}S$ and a given curve $s : (-1, 1) \to S$ such that $s'(0) = t$, we can define

$$K(n, t) = \frac{\partial^2 x}{\partial s^2} \cdot n$$

As the notation suggests, K is independent of choice of s. Intuitively, $K(n, t)$ measures how much the surfaces curves towards n as you travel in the direction of t from the point p.

We then define the mean curvature H at a normal vector n as follows:

$$H(n) = \frac{1}{2}(\max_t K(n, t) + \min_t K(n, t))$$

Definition S as defined above is a minimal surface if the mean curvature of S vanishes identically, that is, $H(n) = 0$ for all $n \in T^{\perp}S$.

Remark 1 Determining whether or not a surface is minimal comes down to an issue of solving second-order differential equations. These equations become much simpler when using isothermal parameters, i.e. the map $x(u, v)$ is conformal.

Remark 2 Many results for minimal surfaces hold instead for generalized minimal surfaces, in which an isolated set of branch points are allowed. Branch points correspond to singularities of the parametrization, where $x(u, v)$ is not an immersion. The behavior of the surface near these points corresponds to the behavior of singular points of complex analytic functions.

Remark 3 The definition above is based on parametrization by a domain D of $\mathbb{R}^2$. However, the property of being an immersion and the definition of H are invariant under a conformal change-of-parameter. Thus it is possible to generalize the definition above to a surface parametrized (conformally) by a Riemann surface.

Remark 4 For the rest of this paper, unless stated otherwise, we will assume that we are in $\mathbb{R}^3$.

3. Classical examples of complete embedded minimal surfaces

Example 1 The plane [fig. 1]. The plane can be described by the set of solutions to the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$. It is the only complete minimal surface in $\mathbb{R}^3$ which is "non-parametric" over the whole (x_1, x_2)-plane, i.e. of the form $x_3 = f(x_1, x_2)$, where f is single-valued. (Note, however, that other minimal surfaces can in general be locally non-parametric.)

Example 2 The catenoid [fig. 2]. The catenoid is the only minimal surface of rotation [Oss.]. It can be described by the set of solutions in $\mathbb{R}^3$ to the equation $x_1^2 + x_2^2 = (\cosh x_3)^2$.

Example 3 The helicoid [fig. 3]. The helicoid is the only ruled minimal surface [Oss.]. It can be described by the set of solutions in $\mathbb{R}^3$ to the equation $x_3 = \tan^{-1}(\frac{x_2}{x_1})$. The helicoid is a singly periodic surface (see §6) since, for example, it is invariant under the group of isometries of $\mathbb{R}^3$ generated by the translation $(x_1, x_2, x_3) \to (x_1, x_2, x_3 + \pi)$.

Example 4 Scherk's surface [fig. 4]. Scherk's surface can be described by the set of solutions to the equation $x_3 = \log \frac{\cos x_2}{\cos x_1}$. If one colors the (x_1, x_2)-plane like a checkerboard where the squares have side length π, with a black square centered at $(0, 0)$, then x_3 is defined for the interior of the black squares. Note x_3 is not defined when $\frac{\cos x_2}{\cos x_1} = 0$, ∞, or is negative. However, at the lattice points (i.e. where $\cos x_1 = \cos x_2 = 0$) we take the closure by allowing limits of solutions, which will in fact give x_3 all possible values at these points. Scherk's surface is a doubly periodic surface (see §6), since it is invariant under the group of isometries of $\mathbb{R}^3$ generated by the translations $(x_1, x_2, x_3) \to (x_1 + 2\pi, x_2, x_3)$ and $(x_1, x_2, x_3) \to (x_1 + \pi, x_2 + \pi, x_3)$.

4. The Enneper-Weierstrass construction

Suppose that we have a conformal parametrization $f : M \to S$, where M is a Riemann surface and S is an immersed minimal surface in $\mathbb{R}^3$. M has a natural orientation which S inherits. Thus there is a continuous choice of unit normal at every point in S which, together with the orientation of S, gives the standard right-hand orientation of $\mathbb{R}^3$. Each unit normal corresponds to a point of $S^2 \subset \mathbb{R}^3$, and this locally defines the Gauss map $G : S \to S^2$. If we identify S^2 with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ through stereographic projection then the composition $g = G \circ f : M \to S^2$ is a meromorphic function on M [Oss.].

It turns out that S can then be recovered from g by the following integral, where μ is a holomorphic 1-form which depends on g:

$$f(p) = \mathrm{Re} \int_{p_0}^{p} (1 - g^2, i(1 + g^2), 2g)\mu$$

A natural question thus arises: given a Riemann surface M and a meromorphic function g on M, when can a similar integration give rise to a minimal surface parametrized by M?

There is in fact only one obstruction. First one must choose a holomorphic 1-form μ with the property that p is a zero (of order $2m$) of μ exactly when p is a pole (of order m) of g. This is always possible. Then the integral above will produce a minimal surface in $\mathbb{R}^3$, provided the integration is well-defined. This will happen whenever integration around any closed curve in M gives zero. Otherwise, the construction will be defined instead on some cover of M [H.-K.].

Examples Here is the Enneper-Weierstrass data for Examples 1-4 above.

	M	g	μ
plane	$\mathbb{C}$	1	dz
catenoid	$\mathbb{C} - \{0\}$	z	$\frac{dz}{z^2}$
helicoid	$\mathbb{C} - \{0\}$	z	$\frac{idz}{z^2}$
Scherk's surface	$\hat{\mathbb{C}} - \{\lambda : \lambda^4 = 1\}$	z	$\frac{dz}{z^4-1}$
bonus surface	$\hat{\mathbb{C}} - \{\lambda : \lambda^4 = 1\}$	z	$\frac{idz}{z^4-1}$

$\hat{\mathbb{C}}$ denotes the Riemann sphere $\mathbb{C} \cup \infty$.

Remark 5 The last three examples do not satisfy the "period condition", i.e. the EW integral is not zero around every closed curve. Thus they are multivalued on the given domains, and well-defined on some cover of M. For example, in the case of the helicoid,

$$\int_\gamma ((z^{-2} - 1), i(z^{-2} + 1), 2z^{-1})idz = (0, 0, -4\pi)$$

where γ is a loop around 0. This corresponds to the translation of $\mathbb{R}^3$ which leaves the helicoid invariant.

Remark 6 The bonus surface on the above list is known as Scherk's singly periodic surface [fig. 5]. It is a conjugate of Scherk's (doubly periodic) surface. If S is a minimal surface given by the Enneper-Weierstrass data $\{g, \mu\}$ on M, then the conjugate of S is given by the data $\{g, i\mu\}$ on M. Note that one set of data may satisfy the period condition without the other doing so, as in the case of the helicoid and the catenoid.

Remark 7 There is a strong relationship between the Gauss map described above and the Gaussian curvature of S. Recall that when S is a surface in $\mathbb{R}^3$, one can define the Gaussian curvature $K(x)$ by choosing a unit normal n to S at x and defining $K(x) = (\max K(n,t) \times \min K(n,t))$. Note that for a minimal surface S one has $K \leq 0$. The integral $\int_S KdA$ is called the total curvature of S. It turns out that this integral is equal to (the negative of) the total spherical area of the image of the Gauss map $G : S \to S^2$, counted with multiplicities. This is in fact the way Gauss originally defined the notion of Gaussian curvature [Oss.]. In particular, when the degree d of G is well defined, we have $\int_S K dA = -4\pi d$.

Note that the Enneper-Weierstrass construction only guarantees an immersed minimal surface. To verify that a particular construction determines an embedded surface is a difficult task. In addition, the immersion is not necessarily complete, as one could easily remove a point from any of the given examples and obtain an incomplete minimal surface. The induced metric on M is $ds = |\mu|(1 + |g|^2)$, so for completeness one needs to check that divergent paths in M have infinite length in this metric.

5. **Complete embedded minimal surfaces of finite total curvature**

For many years the only known examples of complete embedded minimal surfaces were the plane, catenoid, helicoid and periodic surfaces. It was conjectured that other examples of finite total curvature might exist, and the following theorem of Osserman implied that one should look at Enneper-Weierstrass data on Riemann surfaces of finite conformal type for such an example. A surface S is of finite conformal type if it is conformally isomorphic to a compact Riemann surface with a finite number of punctures, or "ends."

Theorem 3 [Oss.] Let M be a complete immersed minimal surface in $\mathbb{R}^3$ with finite total curvature. Then M is of finite conformal type and M can be parametrized by meromorphic data on a compact Riemann surface. More precisely, if $\bar{M}$ denotes the conformal compactification of M (so M is conformally $\bar{M} - \{p_1, \ldots, p_n\}$) then the Weierstrass representation (g, μ) extends meromorphically to $\bar{M}$.

In order to create such an example, then, one might start with a meromorphic function g and holomorphic 1-form μ on a compact Riemann surface M with a chosen set of punctures, such that g and μ extended meromorphically to the punctures. One would verify that g and μ satisfied the zero-pole condition and the period condition, and that the metric on the surface was complete. Then there are two further ways one should adjust the meromorphic function g based on the following two theorems:

Theorem 2 [Sch.] If an end of a properly embedded minimal surface has finite total curvature, it is asymptotic to either the end of a plane or the end of a catenoid. Moreover all these ends are parallel in the sense that their limit normals are parallel.

Theorem 3 (from classical analysis; see [H.-M. 2]) On a complete embedded minimal surface of finite total curvature with genus k and r ends we have $\int_S K dA = 2\pi(2 - 2k - 2r)$.

For a complete minimal surface the Gauss map is a finite covering of S^2, except perhaps for the preimage of a set of points of measure zero. Then for a surface of finite total curvature the sperical area of the image would be $4\pi n$, where n is the degree of the Gauss map. Combining Remark 7 with Theorem 3, one has that the degree of the Gauss map is necessarily $r + k - 1$. This is one condition on g. Also, by Theorem 2, one should ensure that (up to rotation) the extension of g to $\bar{M}$ has only zeros and poles at the punctures $p_1, \ldots p_n$.

Even with such explicit guidelines for construction, until recently no one was able to produce examples of complete embedded minimal surfaces with finite total curvature other than the classical ones (plane and catenoid). In 1981, Costa suggested the meromorphic data $g = cp'$, $\mu = p^{-1}dz$ on a torus with three punctures. Here, p refers to the Weierstrass $p-$ function , which arises in the study of elliptic curves, and c is a constant which ensures that the period condition is satisfied. Also by the choice of c, Costa showed that the minimal surface defined was embedded ouside of some compact set, in addition to satisfying all the previously mentioned conditions. The key step was to understand the structure of the surface near the center. Finally in 1984, Hoffman and Meeks, with the aid of a computer, were able to create images of the surface and prove that Costa's surface is in fact embedded [H.-M. 1][fig. 6].

Since the discovery of Costa's surface there has been an explosion of other surfaces in this category. Hoffman and Meeks have generalized the data for Costa's surface to produce a family of examples with three ends (one planar, two catenoidal) of genus k for $k \geq 2$ [fig. 7]. Together with Callahan and a computer program called MESH, written by J. Hoffman, they have also proved the existence of examples with four ends (two planar, two catenoidal) for every genus [H.-M. 2].

So one possible method of classifying minimal surfaces with finite total curvature is through the set of meromorphic maps on compact surfaces. This may or may not be a fruitful classification method, but it is the only means so far by which examples have been generated. In addition to the recent discovery of these new examples, there have been several classification theorems based on the conformal type of the surface:

Theorem 4 [Sch.] If M is a complete embedded minimal surface in $\mathbb{R}^3$ of finite total curvature with two ends, then M is a catenoid.

Theorem 5 [L.-R.] If M is a complete embedded minimal surface in $\mathbb{R}^3$ of finite total curvature and of genus zero, then M is a plane or a catenoid.

Theorem 6 [Cos.2] If M is a complete embedded minimal surface in $\mathbb{R}^3$ of finite total curvature, of genus one and with three ends, then M is in a one-parameter family of minimal surfaces including the Costa surface [fig. 6].

The next natural question to ask is, which conformal types can be realized as minimal surfaces? The answer is open; however, Hoffman conjectures that complete embedded minimal surfaces of genus k and with r ends must satisfy $r \leq k + 2$. It is not even known whether such realizations would necessarily be of finite total curvature.

More abstractly, what is the space of complete embedded minimal surfaces of finite total curvature? Hoffman and Meeks have a recent result here.

Theorem 7 [Hof.] If complete embedded minimal surfaces of finite total curvature are normalized to have maximum Gaussian curvature 1, occuring at the origin, then the space of such surfaces is compact.

6. **Complete embedded periodic minimal surfaces**

Definition: A minimal surface in $\mathbb{R}^3$ is periodic if it is invariant under a nontrivial discrete subgroup G of the isometries of $\mathbb{R}^3$ which acts freely.

One can study periodic minimal surfaces by instead studying the minimal surfaces of the quotient domain. In the case of the helicoid, for example, although it has nondiscrete symmetry group, we can still take the quotient of $\mathbb{R}^3$ by a discrete subgroup generated by S_θ, where S_θ is an isometry of the helicoid generated by a screw rotation (vertical translation plus rotation) of angle θ. The image of the helicoid in this quotient will then be a complete embedded minimal surface in $\mathbb{R}^3/S_\theta$. In general it is the case that periodic embedded minimal surfaces in $\mathbb{R}^3$ invariant under G correspond exactly to embedded minimal surfaces in $\mathbb{R}/G$. Furthermore, $\mathbb{R}/G$ is finitely covered by one of the following flat 3-manifolds: $\mathbb{R}^3/S_\theta, T^2 \times \mathbb{R}$, or T^3, where T^2, T^3 denote the flat two-torus and three-torus. Thus it is fruitful to work in those three manifolds, hence generating results for singly, doubly and triply periodic surfaces respectively.

Already some results are stronger than for the $\mathbb{R}^3$ case. Whereas in $\mathbb{R}^3$ there is a surface of infinite curvature yet finite conformal type (the helicoid), and the extent of such examples is unknown, in the periodic case we have a direct relationship between topology and curvature (Theorem 8). Theorems 9 and 10 give results for periodic surfaces which mimic results described above for $\mathbb{R}^3$.

<u>Theorem 8</u> [M.-R.] (The finite total curvature theorem) Let M be an embedded minimal surface in a non-simply connected complete flat 3-manifold. Then M has finite topology if and only if it has finite total curvature.

<u>Theorem 9</u> [M.-R.] Let M be a complete minimal surface with finite total curvature in $\mathbb{R}^3/S_\theta$. Then there exists a conformal compactification $\bar{M}$ of M, and meromorphic one-forms $(\frac{dg}{g}, \mu)$ on $\bar{M}$, such that M is parametrized by

$$x(z) = \mathrm{Re} \int (g + \frac{1}{g}, ig - \frac{i}{g}, 2)\mu$$

where $g = \exp(\int \frac{dg}{g})$.

<u>Theorem 10</u> [M.-R.]: Let M be a properly embedded minimal surface in $\mathbb{R} \times T^2$ with finite total curvature $C(M)$. Then, if $\chi(M)$ represents the Euler characteristic of M, we have

$$C(M) = 2\pi\chi(M).$$

(In fact, this theorem generalizes to embeddings in a non-simply connected complete flat 3-manifold, in which case the formula becomes $C(M) = 2\pi(\chi(M) - W(M))$, where $W(M)$ is the "total winding number of the ends of M.")

Theorems 8 and 9 above are related to a host of other theorems about the number and type of ends a complete minimal surface in $\mathbb{R}^3/S_\theta$ can have. In particular, we have Theorem 11. Theorem 12 is a result in $\mathbb{R}^3$ which then follows as a corollary.

<u>Theorem 11</u> [M.-R.] Let M be a complete embedded minimal surface in $\mathbb{R}^3/S_\theta$, topologically an annulus, and not flat. Then M is a helicoid.

<u>Theorem 12</u> [M.-R.] The plane and the helicoid are the only simply connected complete embedded minimal surfaces in $\mathbb{R}^3$ with an infinite symmetry group.

What is known about the possible conformal types of periodic minimal surfaces? Both Scherk's singly and doubly periodic surfaces [figs. 4,5] have infinite genus and only one end. Believe it or not - and I find this hard to believe - this means that they are ambiently isotopic. This is from a result of Frohman and Meeks [F.-M.] and holds for any two properly embedded minimal surfaces of infinite genus and one end. On the other hand, the Riemann example [figs. 8,9] has genus zero and an infinite number of ends, all asymptotic to parallel planes. The following result of Callahan, Hoffman and Meeks is useful here.

Theorem 13 [C.-H.-M.] If M is an embedded periodic minimal surface with discrete symmetry group and more than one end, then in fact M has an infinite number of ends.

Corollary If M is an embedded doubly periodic minimal surface then it will necessarily have only one end.

It is also the case (as follows easily from the definitions) that a triply periodic surface will have only one end. Thus, except for the class of singly periodic surfaces with an infinite number of ends, all embedded periodic minimal surfaces have only one end.

7. **The classification of complete embedded minimal surfaces**

So far in this paper we have discussed two categories of complete embedded minimal surfaces: periodic surfaces and surfaces of finite total curvature. At the first writing of this paper, according to the current available literature, there were no examples outside these two categories. However, it had been conjectured that one could add a handle to the helicoid, a surface which would still have infinite total curvature but no longer be periodic. Shortly afterwards I interviewed Hoffman and asked if there had been any progress on such an example. He instructed me to turn around, and there on his desk was a huge plastic model of a helicoid with a handle. Success had been achieved in 1992 and immortalized in polymers.

Finding the genus one helicoid answered one open question, but there are still many more, all relating somehow to the overall goal of classifying all complete embedded minimal surfaces in $\mathbb{R}^3$.

8. **Open questions and conjectures**

Open question: Are there any complete embedded simply connected minimal surfaces other than the plane and the helicoid? By Theorem 12, such a surface would necessarily have a finite symmetry group.

Nitsche conjecture: a minimal surface that meets every horizontal plane in a Jordan curve is a catenoid. (This has been proved affirmatively for star-shaped curves.)

Related to the above is the finite total curvature conjecture: An annular end of a complete embedded minimal surface in $\mathbb{R}^3$ with at least two ends has finite total curvature.

Open question: which topological types (by genus and number of ends) can be realized as complete embedded minimal surfaces? Pertaining to this is the conjecture mentioned in section 5, that genus $\leq$ (# ends) +2 for surfaces of finite topological type.

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