The Geometry of $\widetilde{\mathrm{SL}_2\mathbb{R}}$

Kevin Hartshort

September 8, 1995

1 Introduction

In a student's introduction to modern differential geometry, there is the an inevitable encounter with the hyperbolic plane $\mathbb{H}^2$ and its geometric structure. One aspect of the study of the geometry of $\mathbb{H}^2$ is the Lie group of isometries which act on this homogeneous space. With this Lie group, a geometric structure can be put on a surface S_g of genus $g \geq 2$ by identifying it with $\Gamma\backslash\mathbb{H}^2$ where Γ acts on $\mathbb{H}^2$ as a discrete subgroup of the orientation preserving isometries. As will be shown, a geometric structure on a manifold lifts to a structure on the tangent bundle of the manifold. In the case of S_g, the structure of the tangent bundle (or more specifically, the *unit* tangent bundle) generalizes to a whole class of 3-manifolds that became especially important through the work of Thurston in the late 70's to find a factorization of 3-manifolds into nice homogeneous spaces.

Before a discussion of the geometry of the unit tangent bundle $U(\mathbb{H}^2)$ of hyperbolic surfaces, however, the notion of geometry should be briefly clarified. In the spirit of Thurston [8] and Scott [5], I shall consider a geometry to be a simply connected space X equipped with a maximal set of isometries G. Specifically, I take from Thurston the following

Definition 1 *A model geometry (X, G) is a simply connected manifold X together with a Lie group G of diffeomorphisms of X such that*

1. *G acts transitively on X and for each $x \in X$, the stabilizer G_x is compact.*
2. *There is at least one compact manifold modeled on (X, G)[1].*
3. *G is maximal under the above conditions.*

Requiring that X be simply connected and G be maximal helps to avoid many redundancies. For example, $\mathbb{R}^2$ with the group of translations or the

[1] A compact manifold M is said to be modeled on (X, G) if there is a discrete subgroup $\Gamma \subset G$ such that the quotient $\Gamma\backslash X$ is homeomorphic to M.

flat torus with its standard metric both have the same "geometry" as $\mathbb{R}^2$ with the entire group of Euclidean isometries. The first condition of the definition allows a (homogeneous) Riemannian metric to be recovered from the group of isometries, while condition (2) ensures a type of existence, since we are generally interested in compact manifolds. Thurston proved that there are exactly eight distinct geometries under this definition. The spaces $\mathbb{R}^3$, S^3, $\mathbb{R}\times S^2$, $\mathbb{H}^3$, and $\mathbb{R}\times\mathbb{H}^2$ together with their standard groups of isometries are fairly intuitive, at least on a superficial level. The geometries Nil and Sol are more complicated Lie group theoretic spaces. The eighth geometry, $\widetilde{\mathrm{SL}_2\mathbb{R}}$, is the focus of this paper as the geometry of unit tangent bundles.

The geometry of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ arises naturally as the geometry of $U(\mathbb{H}^2)$, and so the geometric structure of $\mathbb{H}^2$ will be important in this discussion. Through the paper, I shall use the Poincaré disk model, so $\mathbb{H}^2 = \{z\in\mathbb{C} : |z|<1\}$. The group $\mathrm{Isom}(\mathbb{H}^2)$ of isometries of the disk has two components: the orientation preserving and the orientation reversing isometries. The orientation preserving isometries are given by the Möbius transformations

$$z \longmapsto \frac{az+b}{\bar{b}z+\bar{a}}, \quad |a|^2-|b|^2=1, \quad a,b\in\mathbb{C}.$$

There is a natural identification of $\mathrm{PSL}_2\mathbb{R}$ with these isometries, given as follows. An element $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{PSL}_2\mathbb{R}$ can be considered as an isometry

$$z \longmapsto \frac{\alpha z+\beta}{\gamma z+\delta}$$

on the upper half plane model. The transformation

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \longmapsto \frac{1}{2}\begin{pmatrix}i & 1\\ 1 & i\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}-i & 1\\ 1 & -i\end{pmatrix}$$

gives a matrix of the form $\begin{pmatrix}a & b\\ \bar{b} & \bar{a}\end{pmatrix}$ which represents a Möbius transformation on the unit disk. Clearly, this process is reversible, so there is a natural one to one correspondence between $\mathrm{PSL}_2\mathbb{R}$ and the orientation preserving isometries of the hyperbolic disk. These orientation preserving isometries are generated by maps of three types:

(1) **Rotations:** For example, rotation by θ about the origin in $\mathbb{H}^2$ is given by the map $z\mapsto e^{i\theta}z$.

(2) **Translations along geodesics:** For example, for any $t\in\mathbb{R}$ the map

$$z \mapsto \frac{\frac{e^t+e^{-t}}{2}z + i\frac{e^t-e^{-t}}{2}}{-i\frac{e^t-e^{-t}}{2}z+\frac{e^t+e^{-t}}{2}}$$

translates $\mathbb{H}^2$ along the vertical geodesic through the origin.

(3) **Limit Rotations:** For example, "rotation" about 1 in the circle at infinity is given by $z \mapsto \frac{(1+\alpha i)z+\alpha}{\alpha z+(1-\alpha i)}$ for $\alpha \in \mathbb{R}$.

The other coset of $\text{Isom}(\mathbb{H}^2)$ can be represented by the map $z \mapsto \bar{z}$ composed with a Möbius transformation from above. For details, see [3] or [7]. With this model of $\mathbb{H}^2$ we also have a nice basis for the tangent space at the origin, namely $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

From this model of $\mathbb{H}^2$, I will use the next section to work quickly through the construction of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. The third section will be used to examine some properties of this structure, while the fourth and final section will look at some quotients of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ by subgroups of its group of isometries.

2 The Construction of $\widetilde{\mathbf{SL}_2\mathbb{R}}$

The group $\mathrm{PSL}_2\mathbb{R}$ acts transitively on $\mathbb{H}^2$, since for any point $\omega \in \mathbb{H}^2$, there is an element $\phi \in \text{Isom}(\mathbb{H}^2)$ with $\phi(0) = \omega$, namely

$$\phi(z) = \frac{\frac{1}{|\omega|}z + \frac{\omega}{|\omega|}}{\frac{\bar{\omega}}{|\omega|}z + \frac{1}{|\omega|}}.$$

Further, the stabilizer of a point is easy to deduce. Of the three types of isometries, only the rotations leave a point fixed. In fact, for any $x \in \mathbb{H}^2$, the stablizer group of x in $\mathrm{PSL}_2\mathbb{R}$ is conjugate to the circle group $SO(2)$.

With this in mind, we bring our focus to the tangent bundle $T\mathbb{H}^2$. On $T(T\mathbb{H}^2)$, there is a clear notion of "vertical" - the tangent to the fibre over a point. The horizontal connexion on $T(T\mathbb{H}^2)$ is retrieved from the metric on $\mathbb{H}^2$. Specifically, one can choose a point $v \in T\mathbb{H}^2$ over $x \in \mathbb{H}^2$. Any geodesic through x lifts to a unique path through v by way of parallel translation of tangent vectors along the geodesic. The union of all such paths through v forms a surface in $T\mathbb{H}^2$. The tangent plane of this surface at v gives the horizontal connexion at v. One significant fact to note is that there are actually no horizontal surfaces in $T\mathbb{H}^2$. That is to say, the horizontal connexion on $T(T\mathbb{H}^2)$ is nowhere integrable, not even locally. This can be seen by way of contradiction. For if there were such an integral surface S through v, then the projection $p : S \longrightarrow \mathbb{H}^2$ would be locally injective. Because $\mathbb{H}^2$ is simply conne! cted, the map $p : S \longrightarrow \mathbb{H}^2$ must actually be an injection, and so for any $x \in \mathbb{H}^2$, S intersects $T_x\mathbb{H}^2$ in at most one point. Now look at some loop ℓ though $p(v)$ in $\mathbb{H}^2$ which traces out a triangle with angles α, β, γ. The image of v under parallel translation along ℓ is again in $T_{p(v)}\mathbb{H}^2$, but has been rotated by a factor of $\pi - (\alpha + \beta + \gamma)$. For any triangle in $\mathbb{H}^2$, we have $\alpha + \beta + \gamma < \pi$. This would imply that S intersects $T_{p(v)}\mathbb{H}^2$ in two points - a contradiction. In this way, we see that the horizontal connexion is not even locally integrable.

With a notion of horizontal and vertical, we get a metric on $T\mathbb{H}^2$ pulled back by the projection $p : T\mathbb{H}^2 \to \mathbb{H}^2$, in which the metric on the fibre is the natural metric on a 2-dimensional vector space. Note that there is some choice in the metric (specifically, some scalar multiple). This will be dealt with more carefully in section 3. Now $\mathrm{PSL}_2\mathbb{R}$ acts on $T\mathbb{H}^2$ by isometries (given by the derivatives of the Möbius transforms). The induced metric on $T\mathbb{H}^2$ restricts to a metric on the unit tangent bundle $U(\mathbb{H}^2)$ on which $\mathrm{PSL}_2\mathbb{R}$ actually acts transitively. The transitive nature of the $\mathrm{PSL}_2\mathbb{R}$ action on $U(\mathbb{H}^2)$ is due to the stabilizer of a point in $\mathbb{H}^2$ being a conjugate of the group of rotations. To map a point v over $x \in \mathbb{H}^2$ to the point v' over x', we compose an isometry $\phi : x \mapsto x'$ with an appropriate rotation isometry ψ about x'. Further, $\mathrm{PSL}_2\mathbb{R}$ acts simply on $U(\mathbb{H}^2)$ (that is, for $\phi \in \mathrm{PSL}_2\mathbb{R}$, if there exists an $x \in! U(\mathbb{H}^2)$ such that $\phi(x) = x$, then ϕ is the identity), which is again clear from the fact that the stabilizer of a point is isomorphic to the circle group. For if v over x were fixed, then x would be fixed and so ϕ must be a rotation, but because v is fixed, ϕ must be the identity. Therefore, after choosing an "identity" point on $U(\mathbb{H}^2)$, one can identify $\mathrm{PSL}_2\mathbb{R}$ with $U(\mathbb{H}^2)$, thus inducing a metric on $\mathrm{PSL}_2\mathbb{R}$. In the disk model, this "identity" point shall be chosen to be the vector $\frac{\partial}{\partial y}$ over the origin. Since $\mathrm{PSL}_2\mathbb{R}$ acted on $U(\mathbb{H}^2)$ on the left, the metric on $\mathrm{PSL}_2\mathbb{R}$ is left invariant. $\mathrm{PSL}_2\mathbb{R}$ is double covered by $\mathrm{SL}_2\mathbb{R}$ and thus there is a unique left invariant metric on $\mathrm{SL}_2\mathbb{R}$ making the covering map a local isometry. Because $\mathrm{PSL}_2\mathbb{R}$ (and hence $\mathrm{SL}_2\mathbb{R}$) is a circle bundle over $\mathbb{H}^2$, it has the homotopy type of the solid torus and so has a universal cover $\widetilde{\mathrm{PSL}_2\mathbb{R}} = \widetilde{\mathrm{SL}_2\mathbb{R}}$, which inherits a metric pulled back via the covering map[2].

In studying the geometric structure of $\widetilde{\mathrm{SL}_2\mathbb{R}}$, it will be alternatingly more convenient to look at $\widetilde{\mathrm{SL}_2\mathbb{R}}$, $\mathrm{SL}_2\mathbb{R}$, or $\mathrm{PSL}_2\mathbb{R}$ for various reasons. The simply connected $\widetilde{\mathrm{SL}_2\mathbb{R}}$ has the advantage of conforming to the definition for a geometry. $\mathrm{SL}_2\mathbb{R}$ is easily represented as a subgroup of the matrix group $\mathrm{GL}_2\mathbb{R}$, while $\mathrm{PSL}_2\mathbb{R}$ is easily represented as either $U(\mathbb{H}^2)$ or as the group of orientation preserving isometries of $\mathbb{H}^2$. But since the three spaces are locally isometric, most results in the next section will apply to all three spaces equally.

3 The Structure of $\widetilde{\mathrm{SL}_2\mathbb{R}}$

As the universal cover of a Lie group, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is certainly a Lie group itself. In fact, Dubrovin, Fomenko and Novikov [1] give an interesting description of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ as the group of transformations of $\mathbb{R}$ of the form

$$x \longmapsto x + 2\pi a + \frac{1}{i} ln \frac{1 - ze^{-ix}}{1 - \bar{z}e^{ix}},$$

[2]Even though the geometric structure of this simply connected Lie group comes naturally from $\mathrm{PSL}_2\mathbb{R}$, the more common notation is $\widetilde{\mathrm{SL}_2\mathbb{R}}$.

where $a \in \mathbb{R}$, $z \in \mathbb{C}$. Clearly, if $x \mapsto y$, then $x + 2\pi n \mapsto y + 2\pi n$ for any $n \in \mathbb{Z}$ so that $\mathbb{Z}$ is contained in the central subgroup of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. In fact, $\mathbb{Z}$ is the central subgroup of $\widetilde{\mathrm{SL}_2\mathbb{R}}$, but that will be shown later. Note that $\mathbb{Z} \subset \mathbb{R} \subset \widetilde{\mathrm{SL}_2\mathbb{R}}$, where $\mathbb{R}$ is the set of transformations of the form $x \mapsto x + 2\pi a$. The subgroup $\mathbb{R}$ is *not* central and yet has infinite intersection with the central subgroup. This cannot happen for a Lie subgroup of $\mathrm{GL}_N\mathbb{R}$ (see [1]), and so $\widetilde{\mathrm{SL}_2\mathbb{R}}$ does not have a faithful representation. That is to say, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is not a matrix group.

Fortunately, $\mathrm{SL}_2\mathbb{R}$ is a nice matrix group and so by using the basic fact of the exponential map

$$\det(\exp A) = \exp(\mathrm{tr} A),$$

it is easy to compute that the Lie algebra of $\mathrm{SL}_2\mathbb{R}$ is $\mathfrak{sl}_2\mathbb{R} = \{A \in \mathfrak{gl}_2\mathbb{R} | \mathrm{tr} A = 0\}$. Because of the local isomorphisms, $\mathrm{PSL}_2\mathbb{R}$ and $\widetilde{\mathrm{SL}_2\mathbb{R}}$ also have $\mathfrak{sl}_2\mathbb{R}$ as their Lie algebra. Using the metric on $\mathrm{PSL}_2\mathbb{R}$, we can then define the bracket operation and inner product on $\mathfrak{sl}_2\mathbb{R}$.

With this induced structure on $\mathfrak{sl}_2\mathbb{R}$, I appeal to Milnor's work on the curvature of Lie groups [2], where he makes use of the cross product $u \times v = w$, where $\|w\| = \langle u,u\rangle \langle v,v\rangle - \langle u,v\rangle^2$ and w is orthogonal to both u and v. Because $\mathfrak{sl}_2\mathbb{R}$ is 3-dimensional, there are two choices for w satisfying these conditions. Given an orientation on $\mathfrak{sl}_2\mathbb{R}$, w is chosen so that the ordered triple (u, v, w) is positively oriented. If $\frac{\partial}{\partial f}$ is tangent to the fibre of $\widetilde{\mathrm{SL}_2\mathbb{R}}$, then the orientation for $\mathfrak{sl}_2\mathbb{R}$ will be chosen so that $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial f}\right)$ is positively oriented (equivalently, we can say that $\left(\frac{\partial}{\partial f}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is positively oriented). Utilizing the cross product, Milnor states and proves the following lemma.

Lemma 1 *Let G be a 3-dimensional connected Lie group with left-invariant metric. Choose an orientation for the Lie algebra $\mathfrak{g}$ so that the cross product is defined. Then the bracket operation in $\mathfrak{g}$ is related to the cross product by the formula*

$$[u, v] = L(u \times v),$$

where $L : \mathfrak{g} \to \mathfrak{g}$ is a uniquely defined linear mapping. The Lie group G is unimodular if and only if L is self-adjoint.

If L is self-adjoint (as in the case of $\mathfrak{sl}_2\mathbb{R}$, since $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is unimodular), we can find an orthonormal basis of eigenvectors e_1, e_2, e_3 and real numbers $\lambda_1, \lambda_2, \lambda_3$ such that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2.$$

Modulo a change in sign of the eigenvectors, these λ_i are clearly well-defined. But a change of sign in e_1 will change the sign in λ_1, λ_2, *and* λ_3. With this, curvatures are easy to compute using the curvature form $R_{XY}Z$ and the definition for the Ricci curvature:

$$r(x) = \sum_i \kappa(x, e_i) = \sum_i \langle R_{x e_i} x, e_i \rangle .$$

I summarize the results in the following theorem, proved again in [2].

Theorem 1 *The orthonormal basis e_1, e_2, e_3 diagonalizes the Ricci quadratic form, and the principal Ricci curvatures (ie: the eigenvalues) are given by*

$$r(e_1) = 2\mu_2\mu_3, \quad r(e_2) = 2\mu_1\mu_3, \quad r(e_3) = 2\mu_1\mu_2,$$

where $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$. In particular, the scalar curvature is

$$\rho = \sum_i r(e_i) = 2(\mu_2\mu_3 + \mu_1\mu_2 + \mu_1\mu_3)$$

and the sectional curvature is

$$\kappa(u, v) = \|u \times v\|^2 \rho/2 - r(u \times v).$$

This theorem follows as an easy exercise from the definition

$$R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z,$$

and the formula for the covariant derivative of left invariant vector fields

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(\langle [X,Y], Z \rangle + \langle [Y,Z], X \rangle + \langle [Z,X], Y \rangle \right).$$

To give an explicit computation of the curvatures for $\widetilde{\mathrm{SL}_2\mathbb{R}}$, we need an orthonormal basis for $\mathfrak{sl}_2\mathbb{R}$. This will be easier if we focus our attention to a set of three perpendicular geodesics in $\mathrm{PSL}_2\mathbb{R}$ through the identity element (as identified earlier with a vector over the origin of $\mathbb{H}^2$). These perpendicular geodesic paths can be seen as the path through the fibre and the horizontal lifts of the geodesics in $\mathbb{H}^2$ with derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ at the origin. The path through the fibre is easily given by $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ which lifts to $\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ in $\mathfrak{sl}_2\mathbb{R}$. There is some choice in the length of the fibre, but if we consider the fiber to have length 2π, then $\alpha_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has unit length. The desired pair of perpendicular geodesic paths in $\mathbb{H}^2$ is given by the image! of the origin under the maps

$$z \mapsto \frac{\frac{e^t + e^{-t}}{2} z + i \frac{e^t - e^{-t}}{2}}{-i \frac{e^t - e^{-t}}{2} z + \frac{e^t + e^{-t}}{2}}, \quad z \mapsto \frac{\frac{e^t + e^{-t}}{2} z + \frac{e^t - e^{-t}}{2}}{\frac{e^t - e^{-t}}{2} z + \frac{e^t + e^{-t}}{2}}$$

for any $t \in \mathbb{R}$. Using the identification of $\mathrm{PSL}_2\mathbb{R}$ with $\mathrm{Isom}(\mathbb{H}^2)$, we can lift these to mutually perpendicular paths

$$t \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \frac{e^t+e^{-t}}{2} & \frac{e^t-e^{-t}}{2} \\ \frac{e^t-e^{-t}}{2} & \frac{e^t+e^{-t}}{2} \end{pmatrix}$$

in $\mathrm{PSL}_2\mathbb{R}$. These in turn lift to vectors $\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$ and $\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ in $\mathfrak{sl}_2\mathbb{R}$, which give as the remaining two vectors of the orthonormal basis $\alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. With this orthonormal basis, it is easy to verify that the unique transformation $L(u \times v) = [u, v]$ is given by $L(A) = A^t$, and so this basis actually gives an orthonormal basis of eigenvectors with

$$[\alpha_1, \alpha_2] = \alpha_3, \quad [\alpha_2, \alpha_3] = -\alpha_1, \quad [\alpha_3, \alpha_1] = \alpha_2,$$

$$\lambda_1 = -1, \quad \lambda_2 = \lambda_3 = 1.$$

So we can see that the scalar curvature $\rho = -1$ and that the curvature form R_{XY} is not constant so that $\widetilde{\mathrm{SL}_2\mathbb{R}}$ (for example) is not isometric to $\mathbb{H}^3$.

To close out this section, I would like to quickly describe the isometry group of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. As mentioned before, $\mathrm{PSL}_2\mathbb{R}$ has the homotopy type of the solid torus, so $\pi_1(\mathrm{PSL}_2\mathbb{R}) \cong \mathbb{Z}$ and thus we get the sequence

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{SL}_2\mathbb{R}} \to \mathrm{PSL}_2\mathbb{R} \to 0.$$

The $\mathbb{Z}$ subgroup must be central in $\widetilde{\mathrm{SL}_2\mathbb{R}}$ and in fact must be the entire center since the center of $\mathrm{PSL}_2\mathbb{R}$ is trivial. Note that this sequence does not split. This can be most easily seen by looking at the rotations about a point $x \in \mathbb{H}^2$. As the plane rotates, the S^1 fibres of $\mathrm{PSL}_2\mathbb{R}$ are also rotated, which lifts to a vertical translation along the fibres of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. But while the rotation of a full circle is trivial both in $\mathbb{H}^2$ and in $\mathrm{PSL}_2\mathbb{R}$, it corresponds to the map $x \mapsto x + 2\pi$ in the earlier description of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. So the preimage in $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of the rotation group about a point in $\mathbb{H}^2$ is homeomorphic to $\mathbb{R}$ and thus the sequence cannot split.

But the sequence does give some useful information about the isometry group. The central $\mathbb{Z}$ action extends to an action by $\mathbb{R}$ given by translation along the fibres which projects down to rotations of the $\mathrm{PSL}_2\mathbb{R}$ fibres which, in turn, projects to the identity map on $\mathbb{H}^2$. Thus $\mathbb{R} \hookrightarrow \mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$. Because the metric on $\mathrm{PSL}_2\mathbb{R}$, and hence $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is left invariant, we can see that we also have $\widetilde{\mathrm{SL}_2\mathbb{R}} \hookrightarrow \mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$. Further, if we consider $\mathbb{R}$ and $\widetilde{\mathrm{SL}_2\mathbb{R}}$ as subgroups of $\mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$, then $\mathbb{R} \cap \widetilde{\mathrm{SL}_2\mathbb{R}} \cong \mathbb{Z}$. Now, let Γ be the subgroup of $\mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$

generated by the images of $\mathbb{R}$ and $\widetilde{\mathrm{SL}_2\mathbb{R}}$. Then $\dim\Gamma = 4$. Thurston showed [8] that the stabilizer of a point of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ must be trivial, $SO(2)$ or $SO(3)$, implying that the dimension of $\mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$ is 3, 4 or 6. Thus it is clear that either Γ is the identity component of $\mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$ or $\mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$ is 6 dimensional. But in the second case, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ would have to be symmetric and ! thus isometric to S^3, $\mathbb{H}^3$ or $\mathbb{R}^3$, which cannot happen. So Γ is the identity component and, in fact, it turns out that $\mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$ has exactly two components. The second component is generated by the orientation reversing isometries of $\mathbb{H}^2$. It is interesting to note that since these isometries also reverse the orientation of the fibres over $\mathbb{H}^2$, they actually generate orientation preserving isometries of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. In a sense, these isometries are generated by a rotation about a horizontal geodesic of $\widetilde{\mathrm{SL}_2\mathbb{R}}$. In particular, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ does not admit any orientation reversing isometries.

4 Quotients

In the introduction, we saw that to put a geometric structure on a surface S_g of genus $g \geq 2$, we can find a subgroup $G \subset \mathrm{PSL}_2\mathbb{R}$ with $G \cong \pi_1(S_g)$ such that quotienting by the G action on $\mathbb{H}^2$, we get $G\backslash\mathbb{H}^2$ diffeomorphic to S_g and hence get a geometric structure on S_g. We have also seen that an isometry on $\mathbb{H}^2$ lifts to an isometry on $\mathrm{PSL}_2\mathbb{R}$, and hence to $\widetilde{\mathrm{SL}_2\mathbb{R}}$, by looking at the derivative of the isometry map. So if we let G act on $\mathrm{PSL}_2\mathbb{R}$, then the space $G\backslash\,\mathrm{PSL}_2\mathbb{R}$ gives a natural geometric structure to $U(S_g)$. But then there is a group $\tilde{G}$ acting on $\widetilde{\mathrm{SL}_2\mathbb{R}}$ such that $\tilde{G}$ is an extension by the central $\mathbb{Z}$ subgroup of $\mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 0.$$

In this way we can apply the geometry of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ to $U(S_g)$ for any compact surface of genus $g \geq 2$.

A more interesting case occurs when the action of G on $\mathbb{H}^2$ is not free. For example, let G be a triangle group $\Delta(p,q,r)$. That is to say, G is the group of isometries generated by reflections along the edges of a $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ triangle in $\mathbb{H}^2$. Then the G action includes rotations about the vertices of the triangle, and thus the quotient $G\backslash\mathbb{H}^2$ is not a manifold, but an object called on orbifold. Essentially, an orbifold is a manifold with singularities, and these singularities have a specific structure. Thurston gives the working definition: an n dimensional orbifold is a space locally modeled on $\mathbb{R}^n$ modulo finite faithful group actions. One intuitive example is to look at S_2, thought of as the connected sum of two tori, and to think of the $\mathbb{Z}_2$ action that rotates S_2 in such a way that the two tori of the connected sum are switched. It is fairly evident that this action has two fixed points. ! The quotient of S_2 by this action is an orbifold with two singularities (called cone points in this case).

If this properly discontinuous group action of G is lifted to $\mathrm{PSL}_2\mathbb{R}$ (or to $\widetilde{\mathrm{SL}_2\mathbb{R}}$), then the action on $\mathrm{PSL}_2\mathbb{R}$ is free since even if an isometry ϕ fixes a point $x \in \mathbb{H}^2$, the derivative of ϕ at x will be non-trivial. Thus the quotient space of this action will actually be a smooth manifold. It is a "circle bundle" over the orbifold $Q = G\backslash\mathbb{H}^2$, where the term bundle is generalized in order to account for the fibre(s) over the singularitie(s). Such a circle bundle is called a Seifert fibre space. Thurston's paper [8] states and proves the following theorem.

Theorem 2 *An orbifold Q of finite volume modeled on any of the five fibred geometries ($\mathbb{H}^2\times\mathbb{R}$, $S^2\times\mathbb{R}$, $\widetilde{SL_2\mathbb{R}}$, Nil, Sol) itself fibres over a one or two dimensional orbifold[3].*

Specifically, if Q is modeled on $\mathbb{H}^2\times\mathbb{R}$ or $\widetilde{\mathrm{SL}_2\mathbb{R}}$, then it fibres over a hyperbolic orbifold (an orbifold formed as the quotient $G\backslash\mathbb{H}^2$ for some action G). But in the case of $\mathbb{H}^2\times\mathbb{R}$, Q also fibres over a 1 dimensional orbifold because of the natural identification $\mathrm{Isom}(\mathbb{H}^2\times\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^2)\times \mathrm{Isom}(\mathbb{R})$. Scott [6] gives a nice clarification of the difference between $\widetilde{\mathrm{SL}_2\mathbb{R}}$ and $\mathbb{H}^2\times\mathbb{R}$. Let $G = \Delta(p,q,r)$. Then G can be extended by the central $\mathbb{Z}$ subgroup to $\tilde{G} \subset \mathrm{Isom}(\widetilde{\mathrm{SL}_2\mathbb{R}})$, so that $N = \tilde{G}\backslash\widetilde{\mathrm{SL}_2\mathbb{R}}$ is a Seifert fiber surface over the base orbifold $Q = G\backslash\mathbb{H}^2$. Such a space N will be finitely covered by some circle bundle over a closed surface. This is because any triangle group G contains a closed surface group $\bar{G}$ (ie: $\bar{G}\backslash\mathbb{H}^2$ is a closed 2-manifold). Similarly, G can be extended by $\mathbb{Z}$ to $\tilde{G} \subset \mathrm{Isom}(\mathbb{H}^2\times\mathbb{R})$ so that the same comments! apply. Then N is modeled on $\mathbb{H}^2\times\mathbb{R}$ if and only if this circle bundle is trivial. In this case, there is a fibration of S^1 with fibre $\bar{G}\backslash\mathbb{H}^2$ which factors down to a fibration over a 1-dimensional orbifold with fibre a 2-dimensional orbifold. On the other hand, this is impossible for any such Q modeled on $\widetilde{\mathrm{SL}_2\mathbb{R}}$ since the covering circle bundle would be non-trivial.

In this way, there is a large collection of 3-manifolds with this $\widetilde{\mathrm{SL}_2\mathbb{R}}$ structure. It turns out that if we focus our attention to closed manifolds, Seifert spaces form a *complete* collection of 3-manifolds modeled on $\widetilde{\mathrm{SL}_2\mathbb{R}}$. Specifically I take from Scott the following theorem.

Theorem 3 *Let G be a discrete group of isometries acting freely on $\widetilde{SL_2\mathbb{R}}$ with quotient M. Then the foliation of $\widetilde{SL_2\mathbb{R}}$ by vertical lines descends to a foliation of M and one of the following cases occurs:*

1. *The fibration gives M the structure of a line bundle over a non-closed surface.*

2. *The fibration of M is by lines whose image in M is not closed. In this case, G must be isomorphic to $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$, or the Klein bottle group.*

[3] An orbifold Q is modeled on one of the geometries just as a manifold is: $Q = G\backslash X$, where X is one of the model spaces and G is a discrete subgroup of the isometry group of X that may or may not act freely.

3. The fibration of M is a Seifert fibration.

An explicit description of the Seifert manifolds modeled on $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is given by Raymond and Vasquez [4]. These manifolds form one of the many pieces to the puzzle of classifying 3-manifolds. Thurston made much progress in this puzzle, including proving the following theorems to help piece together some classification ideas:

Theorem 4 *Any 3 dimensional model geometry as defined in the introduction is equivalent to the geometry $(X, Isom(X))$, where X is one of $\mathbb{R}^3$, $\mathbb{H}^3$, S^3, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL_2\mathbb{R}}$, Nil, Sol.*

Theorem 5 *If M is a closed 3-manifold which admits a geometric structure modelled on one of the eight geometries, then the geometry involved is unique.*

Again, the first five geometries of the list $(\mathbb{R}^3, \mathbb{H}^3, S^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R})$ are fairly intuitive. In particular, it is fairly evident that each is a distinct model of 3-manifolds. The Nil geometry is based on the Lie group of matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. It is similar to $\widetilde{\mathrm{SL}_2\mathbb{R}}$ in that it is a twisted line bundle. But while $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is a line bundle over $\mathbb{H}^2$, Nil is a twisted bundle over Euclidean space. The solvegeometry, or Sol, gets its name for being the geometry based on a solvable Lie group. It is the least symmetric of the geometries in that the stabilizer of any point is a finite group. For an example of Nil and Sol geometries, consider a linear map $\phi : T^2 \to T^2$ (ie: a linear map from the plane to itself that preserves the integer lattice). The mapping torus is $M_\phi = (T^2 \times I)/\sim$, with the equivalence relation defined by $(x, 0) \sim (\phi(x), 1)$. The manifold M_ϕ will then (depending on the map ϕ) be modeled on $\mathbb{R}^3$, Nil or Sol.

With these eight geometries in hand, the goal is to take a given 3-manifold M and try to break it down in a reasonably natural way into a union of pieces, each of which admits a geometric structure. A decomposition does exist by breaking M into the connected sum $M_1 \# M_2$ of 3-manifolds, neither of which is homeomorphic to S^3. A manifold is then said to be prime if it cannot be so broken down. It has been shown that a manifold can be decomposed into a finite sum of prime manifolds and that this decomposition is unique, provided that M is orientable. The point of this task (known as the Geometrization Conjecture) is to show that all the pieces actually admit a geometric structure.

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