4-manifolds Which are Homeomorphic but not Diffeomorphic

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1 Introduction

The problem of classifying 4-manifolds is very different in the smooth category than in the topological category because of: (1) the existence of non-smoothable topological 4-manifolds, and (2) the existence of smooth 4-manifolds which are homeomorphic but not diffeomorphic. This paper will outline in an informal way the construction of a family of 4-manifolds which are homeomorphic but not diffeomorphic.

1.1 How to read this paper

The main text is the description of the constructions, which will have some (or many) undefined terms, depending on what background you bring to it. The main text is followed by a section of informal commentary which elaborates on those concepts in the main text which I feel competent to elaborate upon; in the main text italics indicate that more information on the italicized idea can be found in the commentary section; you might want to browse through this latter section as soon as you get lost reading about the constructions.

1.2 Conventions

Unless otherwise specified, a 4-manifold should be taken to be smooth, closed, oriented and simply-connected. The non-simply-connected case is just too complicated to get into. All homology and cohomology should be understood to have integer coefficients. The simply-connected assumption guarantees that H_2 and H^2 are free $\mathbf{Z}$-modules. Some standard manifolds are: B^2 is the 2-ball or disk, T^2 is the 2-torus $S^1 \times S^1$, CP^1 is the complex projective line which is diffeomorphic to S^2 (as the Riemann sphere), and CP^2 is the complex projective plane, a 4-manifold. The symbol "$\approx$" will be used to indicate diffeomorphisms and the symbol "$\cong$" will be used to indicate group isomorphisms.

2 Constructions

The following description is a merger of accounts in [Go-Mr] and [Stern] of the construction of an infinite family of diffeomorphism classes of 4-manifolds in two homeomorphism classes. I first describe some general constructions, then look at a special case which gives the manifolds in question. Knowing that they fall into two homeomorphism classes comes from knowing the *intersection forms* of the manifolds, and knowing that they fall into lots of diffeomorphism classes comes from knowing their *Donaldson invariants.*

2.1 General constructions

$E(1)$ is an *algebraic surface* obtained from CP^2 by *blowing up* 9 points, so topologically it is diffeomorphic to $CP^2\#9\overline{CP^2}$ and has intersection form $(1)\oplus 9(-1)$. It fibers (with singularities) over CP^1 with T^2 as the regular fiber, so it is an *elliptic surface.* There are some choices in the construction of the fibration; the most useful construction has 8 exceptional *fishtail fibers* and 2 exceptional *cusp fibers.*

Construct $E(2)$ as the fiber-sum of two copies of $E(1)$. Briefly, this means: remove a small regular neghborhood of a regular fiber from each copy of $E(1)$ and glue the resulting manifolds together along their boundaries in a way that respects the fibrations on either side. Construct $E(n)$ inductively as the fiber-sum of $E(n-1)$ with $E(1)$. So $E(n)$ fibers over CP^1 with $8n$ fishtail fibers and $2n$ cusp fibers. $E(2)$ has the intersection form $E_8\oplus E_8\oplus 3H$, with *Euler characteristic* 24 and *signature* -16.

Given positive integers $m_1,\ldots,m_r$, $E(n;m_1,\ldots,m_r)$ (as its called in [Stern]), is another elliptic surface, but there is a little more complexity to the fibrations; now there are *multiple fibers* of multiplicities $m_1,\ldots,m_r$. Construct $E(n;m_1)$ from $E(n)$ by performing a *logarithmic transformation* of order m_1 on a regular fiber of $E(n)$ "near" a cusp fiber. Construct $E(n;m_1,\ldots,m_r)$ by performing r log transforms of orders $m_1,\ldots,m_r$ on r regular fibers near a cusp fiber.

2.2 The main examples

The above construction is very general, but not all choices of n and $m_1,\ldots,m_r$ give simply-connected manifolds. The simplest result is that $E(2;p,q)$ is simply-connected when p and q are relatively prime, and these manifolds give the basic examples of 4-manifolds which are homeomorphic but not diffeomorphic. (In [Go-Mr] $E(2)$ is called "the $K3$ surface" and $E(2;p,q)$ is called "V(p,q)".) The main idea behind these manifolds was discovered by Friedman and Morgan [Fr-Mor1] and Okonek and Van de Ven [Ok-VdV] independently. After Donaldson presented his first example of a pair of homeomorphic but not diffeomorphic 4-manifolds, they realized they could construct a whole family of homeomorphic but nondiffeomorphic manifolds; in the notation above, these are the manifolds $E(2;2,q)$ for q odd.

In fact the family of manifolds $E(2;p,q)$, with p and q relatively prime, splits into two homeomorphism classes as follows: Performing log transforms on $E(2)$ does not change its *Euler characteristic* or *signature*. The *classification of intersection forms* gives only two possibilities for the intersection form of a smooth manifold with the same Euler characteristic and signature as $E(2)$: $E_8 \oplus E_8 \oplus 3H$ if the manifold is *spin*, and $3CP^2 \# 19\overline{CP^2}$ otherwise. $E(2;p,q)$ is spin if and only if p and q are both odd. This completely classifies the manifolds $E(2;p,q)$, p, q relatively prime, up to homeomorphism: if p and q are both odd then $E(2;p,q)$ is homeomorphic to $E(2)$, otherwise it is homeomorhic to $3CP^2 \# 19\overline{CP^2}$.

That the manifolds $E(2;p,q)$ fall into lots of different diffeomorphism classes is a special case of a result in [Go-Mr] which will be stated below, and is proved by showing that lots of them have different *Donaldson invariants*. The most systematic way to compute these invariants seems to be that outlined in [Stern], due to Fintushel and Stern. Stern looks at two surgery-like operations on 4-manifolds, *blowups* and *rational blowdowns*, and gives formulas stating how these operations affect the *Donaldson series* of a 4-manifold. He shows how to realize the log transform operation as a series of blowups and rational blowdowns. Putting this together with a computation of the Donaldson series for $E(n)$ (based on another interesting surgery-like construction of $E(n)$) gives explicit formulas for the Donaldson series for $E(n;p,q)$. These formulas are long, messy and meaningless if you (like me) don't really understand the Donaldson invariants so I haven't included them, but it looks like Stern's computations are more generally applicable than Gompf and Mrowka's. Gompf and Mrowka's final statements about which manifolds are not diffeomorphic to which other manifolds was easier for me to understand than Stern's, so I will present that in the next paragraph.

2.3 More diffeomorphism classes in these two homeomorphism classes!

Gompf and Mrowka's result in [Go-Mr] adds an interesting twist to the question of finding manifolds which are homeomorphic but not diffeomorphic. Every example constructed above has the structure of a complex algebraic surface, essentially because all the constructions respected the fibrations, which carry alot of information about the algebraic structures. Gompf and Mrowka add to each of the two homeomorphism classes described above infinitely many mutually nondiffeomorphic 4-manifolds none of which are diffeomorphic to any complex surface or even diffeomorphic to a connected sum of complex surfaces. This is done by considering log transforms on three different regular fibers in $E(2)$, but the trick is that now the fibers come from three different fibrations associated with three different algebraic structures on the underlying manifold $E(2)$. (See 3.3.6 below.)

Stern shows in [Stern] how to construct Gompf and Mrowka's examples using blowups and rational blowdowns, and can thus derive a formula for their Donaldson series which shows which ones are not diffeomorphic. The same argument as above divides them into the two homeomorphism classes according to whether or not they are spin. The manifolds constructed are called $K(p_1, q_1; p_2, q_2; p_3, q_3)$, given 6 positive "multiplicities" p_i, q_i, with each pair (p_i, q_i) relatively prime. Permutations of the indices do not change the diffeomorphism types. Let r be the number of indices $i = 1, 2, 3$ for which $p_i q_i$ is even. $K(p_1, q_1; p_2, q_2; p_3, q_3)$ is spin (and hence homeomorphic to $E(2)$) if and only if $r = 0$ (all of the 6 multiplicities are odd), otherwise it is homeomorphic to $3CP^2 \# 18\overline{CP^2}$. A weak version of the main result in [Go-Mr] is:

Theorem 1 *If*

$$K(p_1, q_1; p_2, q_2; p_3, q_3) \approx K(p_1', q_1'; p_2', q_2'; p_3', q_3')$$

then

$$p_1 q_1 p_2 q_2 p_3 q_3 = p_1' q_1' p_2' q_2' p_3' q_3'$$

Furthermore, $r = r'$ (the number of even multiplicities). If $r \leq 1$ then, up to permutation of the indices $i = 1, 2, 3$, $p_i q_i = p_i' q_i'$. If $r = 2$ then, after permuting the indices so that $p_1 q_1$ is odd and $p_1' q_1'$ is odd, then $p_1 q_1 = p_1' q_1'$.

This result includes the earlier results of [Fr-Mor1], [Ok-VdV] in that $E(2; p, q)$ is just $K(p, q; 1, 1; 1, 1)$; in particular, $E(2; 2, q)$ is diffeomorphic to $E(2; 2, q')$ if and only if $q = q'$. The extra twist comes from a result in [Go-Mr] on which of these manifolds are not complex, a weak form of which is:

Theorem 2 *If the 6 multiplicities are all greater than 1 then $K(p_1, q_2, p_2, q_2, p_3, q_3)$ is not diffeomorphic to a connected sum of complex surfaces.*

2.4 A few other examples

The first example of a pair of homeomorphic but not diffeomorphic 4-manifolds was given by Donaldson in an early application of his gauge-theory-inspired invariants in [Don]. It was $E(1)$ as above and an algebraic surface known as the Dolgachev surface. Elliptic surfaces are just a small class of complex algebraic surfaces, so that people have been interested in constructing examples that are not elliptic surfaces. Wolfgang [Wolf], for example, constructs an example using complete intersection surfaces. Donaldson and Kronheimer [Don-Kr] show that, for given any one of a certain large class of algebraic surfaces there is a way to construct a 4-manifold which is homeomorphic to the surface but not diffeomorphic to any complex surface.

3 Some commentary and elaboration

This section should be read as necessity and interest dictate, but here are the main tangential ideas:

-The way people know that two 4-manifolds are homeomorphic usually depends on a topological 4-dimensional version of the h-cobordism theorem, the smooth higher-dimensional version of which is very powereful in classifying higher-dimensional smooth manifolds. The examples discussed here demonstrate the failure of the h-cobordism theorem in the smooth four-dimensional category.

-The way people know that two homeomorphic 4-manifolds are not diffeomorphic involves diffeomorphism invariants which come from considering *geometric* structures on 4-manifolds. They are based on lots of hard Riemannian geometry much of which was inspired by theoretical physics.

-Most of the manifolds considered are complex algebraic surfaces, as the above invariants are often more readily computed from the perspective of algebraic geometry. The classification of algebraic surfaces is a rich subject in its own right and has to some extent guided the search for examples and the operations used in the constructions.

3.1 The intersection form and how to tell when two 4-manifolds are homeomorphic

This is a quick synopsis of what I learned in [Kirby]. The classical invariant of a 4-manifold M is its *intersection form.* This is the integral, symmetric, unimodular, bilinear form on $H_2(M) \cong \mathbf{Z}^n$ which is Poincaré dual to the cup-product pairing on $H^2(M)$. Given two homology classes $x, y \in H_2(M)$, the intersection form gives an integer $x \cdot y$. Two bilinear forms on $\mathbf{Z}^n$ are the same if a change of basis takes one to the other. Whitehead discovered that the homotopy type of a 4-manifold is completely determined by its intersection form. Freedman then made the remarkable discovery in 1981 that in fact the homeomorphism type is completely determined by the intersection form. (There are some complexities in the non-smooth case).

The intersection form has a concrete interpretation. Every homology class in $H_2(M)$ can be represented by a smoothly embedded, oriented 2-manifold. If $x, y \in H_2(M)$ are represented by 2-manifolds F and G respectively, then F and G can be assumed to intersect transversely in a finite number of points, each with an associated sign coming from the orientations of F, G and M. $x \cdot y$ is exactly the number of points of intersection counted with sign. To calculate $x \cdot x$ you need to count the number of points of *self-intersection* of F; consider F and a copy of F which is isotoped slightly so as to be transverse to F, then count the points of intersection. If you have a tubular neighborhood of F, think of this as pushing F off itself along the fibers of the tubular neighborhood.

Wall had shown in 1965 that two 4-manifolds M_0 and M_1 with the same intersection form are h-cobordant, meaning roughly that there exists a 5-manifold W with $\partial W = M_0 \cup M_1$ and $H_i(W, M_0) \cong H_i(W, M_1) \cong \{0\}$ for $i > 0$. In higher dimensions, the h-cobordism theorem says that h-cobordant manifolds are diffeomorphic, but the 4-dimensional h-cobordism "theorem" was an open question for a long time and turned out to be false due to the examples in this paper. Freedman, however, managed to extend the h-cobordism theorem to the topological four-dimensional category to show that two h-cobordant 4-manifolds are homeomorphic.

Finding out what intersection form a 4-manifold has is not always easy. The intersection form can be described by writing down a unimodular, symmetric, integral matrix in terms of some basis for $H_2(M)$, but even then it may be difficult to tell if two matrices represent the same bilinear form. In certain cases a few invariants of the manifold do completely determine the intersection form. If you diagonalize the intersection form over the rationals, b^+ is the number of positive eigenvalues and b^- is the number of negative eigenvalues. The rank of $H_2(M)$ is $b^+ + b^-$, the *Euler characteristic* of M is $b^+ + b^- + 2$, and the *signature* of M is $b^+ - b^-$. The particular case needed here is that if a smooth manifold M has Euler characteristic 24 and signature -16 then its intersection form is either $E_8 \oplus E_8 \oplus 3H$ (when M is *spin*) or $3(1) \oplus 19(-1)$ (when M is not spin). This means that with respect to some basis the matrix will be block diagonal with the given matrices on the diagonal, where (1) and (-1) are the 1×1 matrices, H is a certain 2×2 matrix called a hyperbolic pair, and E_8 is a mysterious 8×8 matrix that is too weird to get into.

3.2 The Donaldson invariants and how to prove that two 4-manifolds are not diffeomorphic

This is an amalgamation of accounts given in [Fr-Mor2] and [Fr-M-M]. The central tools are polynomial diffeomorphism invariants of 4-manifolds. The *Donaldson polynomials* are a sequence of polynomials $\gamma_n(M)$ associated to a 4-manifold M, and defined for all n "sufficiently large". M needs to be simply-connected and $b^+(M)$ needs to be odd and ≥ 3. For each positive integer n there is a way to construct a principal $SU(2)$-bundle P_n over M (it has second Chern class n). Pick a "generic" Riemannian metric g on M. Now you know what it means for a connection on P_n to be "anti-self-dual". Let $\mathcal{X}$ be the space of all connections on P_n modulo the gauge group, and let $\mathcal{M}$ be the space of anti-self-dual connections modulo the gauge group. These are called *moduli spaces*. $\mathcal{M}$ is a finite-dimensional, oriented subspace of $\mathcal{X}$, and you would like to consider the homology class which $\mathcal{M}$ represents in $H_*(\mathcal{X})$, but $\mathcal{M}$ is usually not compact. Difficult mathmatics succeeds in canonically compactifying $\mathcal{M}$ so that, for n sufficiently large, there is a well-defined homology class $[\mathcal{M}] \in H_d(\mathcal{X})$, where d is the dimension of $\mathcal{M}$.

When b^+ is odd, d is even, which is needed for what follows. There is a useful map $\mu : H_2(M) \to H^2(\mathcal{X})$. Suppose $\alpha_1, \ldots, \alpha_{d/2} \in H_2(M)$. Then $\mu(\alpha_1) \cup \ldots \cup \mu(\alpha_{d/2}) \in H^d(\mathcal{X})$ and we can evaluate it on $[\mathcal{M}]$ to get a real number. This number is the definition of $\gamma_n(\alpha_1, \ldots, \alpha_{d/2})$.

The good thing is that these polynomials are diffeomorphism invariants (when defined) and seem to be very good at distinguishing manifolds which are homeomorphic but not diffeomorphic. The bad thing is that they are in general very hard to compute. Thus a great deal of the research energy has gone into identifying special cases where certain tricks make the computations feasible. The central simplifying tool works for algebraic surfaces with Kähler metrics, hence the obsession with algebraic surfaces. In this case the moduli spaces you want to know about are closely related to the moduli spaces of certain types of complex vector bundles, a subject which algebraic geometers know about and have studied for a while.

With this machinery, for a restricted class of algebraic surfaces it is known that the *canonical class*, a special cohomology class coming from the algebraic structure, gives most of the same information that the Donaldson polynomials give (see [Fr-M-M]). A breakthrough of sorts came when Kronheimer and Mrowka [Kr-Mr] decided to put all the Donaldson polynomials together into a formal power series, the *Donaldson series*, and study its asymptotics. This series is shown to be determined by the intersection form, p nonzero rational numbers, and p cohomology classes in $H^2(M)$ called *basic classes*. These basic classes can then be used as diffeomorphism invariants.

3.3 Complex algebraic surfaces, elliptic surfaces and fibrations

A good reference for this is [Fr-Mor1]. A *complex algebraic surface* is a 4-manifold which arises as the intersection of the zero sets of a collection of homogeneous polynomials in some $\mathbf{CP}^n$. (An *algebraic curve* is a 2-manifold which is the intersection of zero sets of polynomials.) CP^2 is the simplest complex algebraic surface and CP^1 is the simplest algebraic curve. Some complex algebraic surfaces can be given the structure of a *fibration* over CP^1 with generic fiber a torus; in this context this means that there is a projection from the surface to CP^1 which is locally trivial almost everywhere. Thus over small B^2-neighborhoods around most points in CP^1 , the fibration looks like the projection $T^2 \times B^2 \to B^2$, but a finite number of points are allowed where local triviality fails; the fibers over these points are called *exceptional fibers*. The fibration has to be compatible with the algebraic structure, so that the fibers are algebraic curves. The key fact is that a nonsingular cubic curve (the zero set of a third-degree homeogeneous polynomial in CP^2) is diffeomorphic to T^2. An algebraic surface which can be given such a fiber structure is called an *elliptic surface*.

3.3.1 Exceptional fibers

The types of exceptional fibers that can occur have been the subject of alot of study; see [H-K-K] for an overview. The ones of interest here are *fishtails*, *cusps* and *multiple fibers*. A fishtail fiber is an immersed sphere S^2 with one point of transverse self-intersection and a cusp fiber is an immersed S^2 with one point of self-intersection which locally looks like a "cone on a trefoil knot". Now note that all the regular fibers are homologous. A multiple fiber of multiplicity m is a torus fiber which is homologous to m regular fibers.

3.3.2 Blowups

Many types of singularities in algebraic geometry are resolved by blowing up points. Topologically, blowing up a point p in a 4-manifold M is the same as removing a small ball around p leaving an S^3 boundary, removing a small ball from a copy of CP^2 , reversing the orientation of the CP^2 , and gluing it in to fill the hole in M. The idea behind the reversal of orientation is that you have to turn CP^2 inside out to make it match up with the complex structure on M and "fit" in the hole. Thus the new manifold is $M\#\overline{CP^2}$ where $\overline{CP^2}$ is CP^2 with its orientation reversed. If a family of curves (which are 2-manifolds) in M all meet transversely at p, blowing up p gives a new manifold in which the curves extend across the $\overline{CP^2}$ naturally so as to completely miss each other and pass through every point in the $\overline{CP^2}$. Note also that CP^2 has a natural CP^1 in it with self-intersection 1 and $\overline{CP^2}$ has a CP^1 with self-intersection -1. Every curve passing through p will end up intersecting the CP^1 in the $\overline{CP^2}$ once.

3.3.3 The construction of E(1) and its fiber structure

This comes from [Go-Mr]. Remember that every nonsingular cubic curve in CP^2 is diffeomorphic to T^2. Also, two cubic curves in CP^2 meet generically in 9 points, so that picking 9 points in CP^2 gives a family of cubic curves all meeting in those 9 points but mutually disjoint everywhere else, filling up all of CP^2 . Depending on how the 9 points are picked, more or less of the curves will be singular, but the choice can be done in such a way that 8 of them are fishtails, 2 of them are cusps, and the rest are nonsingular. Pick 9 such points in CP^2 , take note of the family of cubics, and then blow up each of the 9 points, keeping track of the curves; now they don't intersect at all. The new manifold, $CP^2\#9\overline{CP^2}$ is now foliated by (mostly) tori, and each leaf hits each CP^1 coming from each $\overline{CP^2}$ exactly once. Thus any of these CP^1 's can be taken to be the base space for the fibration via projection parallel to the curves. The curves are the fibers. Note that this fibration thus has 9 disjoint sections, the 9 CP^1 's, each with self-intersection -1.

3.3.4 Logarithmic transformation

This also comes from [Go-Mr]. In an elliptic curve, a regular neighborhood of a regular fiber is diffeomorphic, in a way compatible with the fibration, to $T^2 \times B^2$. The operation of removing such a neighborhood and gluing it back in by a self-diffeomorphism of its boundary (which is diffeomorphic to $T^2 \times S^1 \approx S^1 \times S^1 \times S^1$) is a *logarithmic transformation* on the chosen fiber. This should be done in such a way as to respect the fibre structure on $T^2 \times S^1$; the *order* m of the transform is the number of times the base S^1-factor wraps around itself. The result is that the core $T^2 \times 0$ of the $T^2 \times B^2$ becomes a multiple fiber of multiplicity m. The diffeomorphism type of the resulting manifold is completely determined by m if the log transform happens near a cusp fiber; somehow the cusp fiber adds just enough of a twist to the fiber structure to eliminate any ambiguity..

3.3.5 Rational blowdowns

This comes from [Stern]. For each positive integer p there is a *rational blowdown* operation. Given a 4-manifold M, the first step is to find a submanifold diffeomorphic to a certain 4-manifold-with-boundary called $C(p)$, leaving the lens space $L(p^2, p-1)$ as boundary, and gluing in another 4-manifold called $B(p)$ which has the same lens space as boundary. $C(p)$ is a regular neighborhood of a certain configuration of $(p-1)$ embedded S^2's with specified self-intersections and pairwise intersections, so that finding $C(p)$ can boil down to finding $(p-1)$ homology classes with the given intersection configuration and somehow knowing that those homology classes can be represented by embedded S^2's. Stern then shows that performing a log transform of order p on a regular fiber in $E(n)$ is the same as blowing up $(p-1)$ points, realizing that this new manifold contains a $C(p)$ and replacing that $C(p)$ by $B(p)$. The point of this construction is that Fintushel and Stern know how to compute its effect on the Donaldson series of a 4-manifold, as well as the effect of a blowup. Blowups and rational blowdowns then give lots of 4-manifolds with computable Donaldson invariants.

3.3.6 A little more about Gompf and Mrowka's constructions

In [Go-Mr], Gompf and Mrowka give a twisted construction of $E(2)$ from two copies of $E(1)$: Remove a regular neighborhood of a regular fiber from each copy of $E(1)$ and glue the two manifolds together along their boundaries,as before, but now ignore the fact that the boundaries have the fiber structure of $T^2 \times S^1$ and look at the boundaries just as $S^1 \times S^1 \times S^1$. A cyclic permutation of the 3 S^1 factors while gluing the two manifolds together guarantees that the fiber structure on one side is incompatible with that on the other side. If this is done near a cusp fiber, the resulting manifold is still diffeomorphic to $E(2)$, but there are two different fiber structures, one for each copy of $E(1)$. In fact there are 3 different fiber structures, which correspond to three different incompatible algebraic structures on the underlying manifold. 3 pairs of regular fibers, one from each fiber structure, can be found, each pair near to a cusp fiber from that fiber structure. The 6 regular neighborhoods and regular neighborhoods of the 3 cusp fibers are all disjoint, so that log transforms can be performed on all 6 fibers. The 6 multiplicities of these log transforms are the numbers p_i, q_i in section 2.3. This sufficiently scrambles the 3 fiber structures so as to produce manifolds not diffeomorphic to any complex surface, but it keeps the homeomorphism type under control.

4 Conclusion

I wish I could have made this paper more coherent but the topic naturally brings up alot of material from different fields. Much of it is hopelessly out of date because of the new Seiberg-Witten invariants. See Blake Mellor's paper.

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