Projective Connections on Riemann Surfaces

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1 Introduction

There are two principal objectives of this paper. One is to explain what the classical Schwarzian derivative is in a way that makes sense. The other has more to do with differential geometry per se. In [G] there are purely sheaf theoretic definitions of "affine connections" and "projective connections" on Riemann surfaces. The language commonly used to discuss "projective and affine structures" (in, e.g. [T1]) employs words like "holonomy" and "development" which have strong differential geometric connotations. However, it is not at all clear at first glance, at least to a novice such as myself, whether they really mean the same things or if they are just being overused. The eventual conclusion is that these words *do* mean the same things in both fields. This paper aims to outline how the definitions of "holonomy" and "development" of projective structures given, for example, in [T1] are special cases of the corresponding differential geometry notions given, for example, in [KN]. To this end, we provide a differential geometric interpretation of the projective connections defined in [G]. To properly motivate this, we first set ourselves the task of interpreting [G]'s definition of affine connection.

Added in Proof: After this paper was written, the delightful book [Ko] came to my attention. This book contains, among other things, thorough generalizations of the ideas and methods in this paper, but it apparently does not work out the connection with the classical Schwarzian derivative.

2 The Affine Case

2.1 Generalized Affine Connections

Given manifolds M and N and smooth $f, g : M \to N$, we say that f and g have the same k-jet at $x_0 \in M$ if

$$f(x) = g(x) + o(x - x_0)^k$$

in some, hence any, coordinates near x_0 and $f(x_0)$. This is the same as saying that the degree k Taylor approximations of f and g in these coordinates are identical. The k-jet of f at x will be denoted $j_x^k f$.

For a manifold M of dimension d, we define the affine bundle of bases as

$$\mathcal{A}(\mathcal{M}) = \{|_{\S}^{\infty}\psi \text{ such that } \psi : \mathbb{R}^{\Gamma} \to \mathcal{M} \text{ is nonsingular at } \S\}.$$

This is a principal bundle over M since the group $A(d)$ of affine isomorphisms of $\mathbb{R}^d$ acts freely on $\mathcal{A}(\mathcal{M})$ on the right by

$$(j^1_x\psi)g = j^1_{g^{-1}(x)}(\psi \circ g).$$

The quotient is M and the projection is given by $j^1_x\psi \mapsto \psi(x)$.

In [KN], $\mathcal{A}(\mathcal{M})$ is taken to consist of pairs (b, t) where b is a linear isomorphism of $\mathbb{R}^d$ with some tangent space $T_m M$ and t is a vector in $T_m M$. This pair is thought of as an *affine* isomorphism of $\mathbb{R}^d$ with $T_m M$, namely $x \mapsto b(x) + t$. The definition above is equivalent to this by the correspondence $j^1_x\psi \mapsto (T_x\psi, -T_x\psi(x))$.

Following [KN], a *generalized* affine connection on M is a connection in $\mathcal{A}(\mathcal{M})$. In other words, it is an $\mathfrak{a}(d)$-valued 1-form ϕ on $\mathcal{A}(\mathcal{M})$ such that

- $X = \phi(\bar{X})$ for any $X \in \mathfrak{a}(d)$, and
- $\phi \circ TR_g = \mathrm{Ad}g^{-1} \circ \phi$ for any $g \in A(d)$.

Here $\bar{X}$ denotes the vertical vector field corresponding to X under the representation of $\mathfrak{a}(d)$ into vector fields on $\mathcal{A}(\mathcal{M})$ induced by the action of $A(d)$, TR_g is the derivative of right multiplication by g on $\mathcal{A}(\mathcal{M})$, and Ad is the adjoint.

Let $\{U_\alpha\}$ be an open cover of M with charts $\{x_\alpha : U_\alpha \to \mathbb{R}^d\}$. Then we can define sections $\mathcal{X}_\alpha : U_\alpha \to \mathcal{A}(\mathcal{M})$ in the obvious way, namely $\mathcal{X}_\alpha(m) = j^1_{x_\alpha(m)} x_\alpha^{-1}$. Then ϕ is determined by the properties listed above and the pulled down forms $\mathcal{X}^*_\alpha\phi$.

Let the transition functions be denoted $f_{\alpha\beta}$, so that $x_\alpha = f_{\alpha\beta}(x_\beta)$ holds on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Further, define $A_{\alpha\beta} : x_\beta(U_{\alpha\beta}) \to A(d)$ by $j^1_x A_{\alpha\beta}(x) = j^1_x f_{\alpha\beta}$. In English, $A_{\alpha\beta}(x)$ is the unique affine isomorphism of $\mathbb{R}^d$ whose value and first derivative at x agree with those of $f_{\alpha\beta}$, i.e. it is the osculating affine map to $f_{\alpha\beta}$ at x. With this notation, we have, on $U_{\alpha\beta}$ that

$$\mathcal{X}_\beta = R_{A_{\alpha\beta}\circ x_\beta} \circ \mathcal{X}_\alpha.$$

It is a potentially entertaining but straightforward exercise in differentiation to verify that

$$\mathcal{X}^*_\beta\phi = dA_{\alpha\beta}(x_\beta) + \mathrm{Ad}A_{\alpha\beta}(x_\beta)^{-1} \circ \mathcal{X}^*_\alpha\phi$$

where $dA_{\alpha\beta}(x_\beta)$ is the derivative of $A_{\alpha\beta} \circ x_\beta$ viewed as an $\mathfrak{a}(d)$-valued 1-form. Conversely, a collection $\{\phi_\alpha\}$ of $\mathfrak{a}(d)$-valued 1-forms with ϕ_α defined on $x_\alpha(U_\alpha)$, satisfying

$$\phi_\beta \circ Tx_\beta = dA_{\alpha\beta}(x_\beta) + \phi_\alpha \circ Tx_\alpha \tag{1}$$

determines a generalized affine connection on M.

2.2 Affine Connections

Since $A(d)$ acts effectively on $\mathbb{R}^d$, the induced representation of $\mathfrak{a}(d)$ into the algebra of vector fields on R^d is faithful. Thus we can think of an element $X \in \mathfrak{a}(d)$ as the corresponding vector field. The time t flow along this vector field is given by the action of $\exp(tX)$, and X can be profitably thought of as an "infinitesimal affine map."

Suppressing the subscripts from the previous section, suppose that f is a diffeomorphism from a open subset U of $\mathbb{R}^d$ to another subset of $\mathbb{R}^d$ and $A : U \to A(d)$ is the family of osculating affine maps. Then dA is an $\mathfrak{a}(d)$-valued 1-form on U, but more can be said. In fact, if v_p is a vector at $p \in U$, then $dA(v_p)$ is an infinitesimal affine map *vanishing at* p. To see this, note that for $x, q \in U$, we have $A(q)(x) = f(x) + o(x - q)$ from the definition of A. In particular,

$$A(p + tv_p)(p) = f(p) + o(t) = A(p)(p) + o(t).$$

Then $A(p)^{-1} \circ A(p + tv_p)(p) = p + o(t)$, and this is precisely the claim.

We can use this information to describe the property which distinguishes affine connections from generalized affine connections. First we define a sub-bundle $B \subset M \times \mathfrak{a}(d)$ as follows. Let B_α be the (trivial) bundle over U_α whose fiber over m is the subspace of $\mathfrak{a}(d)$ which vanishes at $x_\alpha(m)$. Then

$$B = \left. \bigcup_\alpha B_\alpha \middle/ \sim \right.$$

where $(m, X) \in B_\alpha \sim (m, Y) \in B_\beta$ if $X = (A_{\alpha\beta}(q))_* Y$.

We have shown that $dA_{\alpha\beta}(x_\beta)$ is a section of B over $U_{\alpha\beta}$. With this in mind, we define an affine connection to be a generalized affine connection ϕ such that every $\mathcal{X}_\alpha^* \phi$ takes values in B. (Compare [KN].)

This bundle B may seem a bit mysterious at first, but in fact it is precisely $\mathrm{End}(TM)$, the bundle whose fiber over m consists of all linear maps $T_m M \to T_m M$. The chart x_α identifies $T_m M$ with $\mathbb{R}^d$ as affine spaces with $0 \in T_m M$ corresponding to $\mathcal{X}_\alpha(m) \in \mathbb{R}^d$. So the fiber of B over m consists exactly of the infinitesimal *linear* maps $\mathfrak{gl}(T_m M)$, which is canonically $\mathrm{End}(T_m M)$. (Alternatively, it's not hard to see the isomorphism of B with the bundle associated to $\mathcal{A}(\mathcal{M})$ through the adjoint action on $\mathfrak{gl}(d)$, thought of as a subspace of $\mathfrak{a}(d)$.)

It is also important to note that $A = \{dA_{\alpha\beta}(x_\beta)\}$ represents a 1-cocycle class in $H^1(M, \mathrm{End}(TM) \otimes T^*M)$ which is canonically associated with M. The cocycle condition

$$0 = (\delta A)_{\alpha\beta\gamma} = dA_{\alpha\beta}(x_\beta) + dA_{\beta\gamma}(x_\gamma) + dA_{\gamma\alpha}(x_\alpha)$$

is exactly the infinitesimal form of the more obvious identity

$$\mathrm{id} = A_{\alpha\beta}(x_\beta) \circ A_{\beta\gamma}(x_\gamma) \circ A_{\gamma\alpha}(x_\alpha).$$

The fact that the cohomology class of A is independent of the coordinate covering $\{U_\alpha, x_\alpha\}$ is left to the reader.

In view of equation 1 of the previous section, an affine connection is a 0-cochain $\phi \in C^0(M, \mathrm{End}(TM) \otimes T^*M)$ such that $\delta\phi = A$. This is essentially the definition of affine connection which appears in [G]. Note that the existence of an affine connection is equivalent to the vanishing of A in cohomology. In the smooth category, $H^1(M, V)$ is trivial for any vector bundle V because of the existence of partitions of unity, thus confirming the fact that every smooth manifold admits an affine connection. In more rigid categories, the question of existence is not so trivial. For example, if a compact Riemann surface admits an affine connection (we are implicitly repeating all of the calculations above replacing "real" with "complex" everywhere and insisting that functions, forms, etc. be holomorphic) then it has genus zero [G].

2.3 Flat Affine Connections

In this section we will show that a *flat* affine connection on M is equivalent to an affine *structure* on M. By an affine structure, we mean an atlas on M compatible with the given one whose transition functions $f_{\alpha\beta}$ are affine maps.

Suppose ϕ is a flat affine connection on M. Choose a point $m \in M$ and think of $\widetilde{M}$, the universal cover, as the space of homotopy classes of paths with initial point m. We define the developing map dev : $\widetilde{M} \to T_m M$ by letting dev(γ) be the image of $0 \in T_{\gamma(1)}M$ under parallel translation along γ to $T_m M$.

Because ϕ is flat, dev(γ) depends only on the homotopy class (with endpoints held fixed) of γ. To see this, let γ_s be a homotopy rel endpoints and $\tilde{\gamma_s}$ the horizontal lifts to $\mathcal{A}(\mathcal{M})$ all with the same initial point. Since ϕ is flat, for each $t \in [0,1]$ there is a horizontal d-dimensional submanifold H_t of $\mathcal{A}(\mathcal{M})$ passing through $\tilde{\gamma_0}(t)$. By compactness of $[0,1]$, for sufficiently small s, the image of $\tilde{\gamma_s}$ will lie entirely in $\bigcup H_t$. Because $\tilde{\gamma_s}(1)$ lies in the fiber over $\gamma_0(1)$, we find that $\tilde{\gamma_0}(1) = \tilde{\gamma_s}(1)$ whenever $0 \leq s \leq s_1$. Repeating this argument finitely many times (compactness of $[0,1]$), we obtain $\tilde{\gamma_1}(1) = \tilde{\gamma_0}(1)$, as required.

Clearly, dev depends on the point m. Suppose $m' \in M$, $\widetilde{M}'$ the universal cover realized as classes of paths starting at m', and dev$'$: $\widetilde{M}' \to T_{m'}M$ devlopment into the tangent space at m'. Choose any path α from m' to m. Then α induces a homeomorphism $\widetilde{M} \to \widetilde{M}'$ over M (by $\gamma \mapsto \gamma\alpha$), and an affine isomorphism $T_m M \to T_{m'}M$ (by parallel translation), making the diagram commute:

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & \widetilde{M}' \\ \downarrow & & \downarrow \\ T_m M & \longrightarrow & T_{m'}M \end{array}$$

Thus dev is well defined up to deck transformation of $\widetilde{M}$ and affine isomorphism of the range. In fact, taking the special case of a loop α at m, we find that the holonomy $h : \Pi_1(M) \to A(d)$ satisfies

$$h(\alpha) \circ \text{dev} = \text{dev} \circ \alpha.$$

There is another essential property of dev, namely that it should be locally injective. Once this is established, then the developing map and holonomy together provide the affine structure on M that we seek — local injectivity provides local homeomorphisms of M with the affine space $T_m M$ and the equation relating dev and h above says precisely that the transition functions are affine. It is left to the reader to convince himself that, conversely, an affine structure determines a flat affine connection, inverting the process described here.

The fact that dev is locally injective is actually obvious once the definition of affine connection is properly digested. If we can show that every $\gamma_0 \in \widetilde{M}$ has a neighborhood U such that dev$(\gamma) \neq$ dev(γ_0) whenever $\gamma \in U - \{\gamma_0\}$, then local injectivity follows immediately from local compactness. We can even assume that γ_0 is the constant curve at m by considering the developing map into $T_{\gamma(1)}M$. So what we need to show is that parallel translation along sufficiently short paths γ is *not* linear, i.e. does not send $0 \in T_m M$ to $0 \in T_{\gamma(1)}M$. This is exactly what the definition of affine connection guarantees. Assume x is a coordinate near m. As mentioned previously, $T_p M$ is identified with $\mathbb{R}^d$ as an affine space by x with $0 \in T_p M$ corresponding to $x(p)$, and parallel translation along γ to $\gamma(t)$ is effected by an affine map $A_t : \mathbb{R}^d \to \mathbb{R}^d$. The definition of affine connection requires that $A_t \circ x(m) = x(m) + o(t)$, i.e. that $x(m)$ is fixed to first order in t. But for parallel translation along γ to be linear would require $A_t \circ x(m) = x \circ \gamma(t)$, and (perhaps after a suitable reparamatrization of γ) $x(m)$ would *not* be fixed to first order in t. Hopefully these words have not obscured the issue too much.

3 Projective Case

For the remainder of the paper, R will denote a Riemann surface, i.e. a connected 1-dimensional complex analytic manifold. Everything will be assumed holomorphic unless explicitly stated otherwise. Some of the subtleties which arise in the passage from the real to the complex setting will be ignored without apology or mention.

Much of this section is handled in greater generality in [Ko], which the interested or perplexed reader is encouraged to consult. Care has been taken to follow the previous section as closely as possible, and it is no accident that passages are copied over verbatim but for the replacement of x's with z's.

3.1 Generalized Projective Connections

Given a Riemann surface R, we define

$$\mathcal{P}(R) = \{j^2_z\psi \text{ such that } \psi : \widehat{\mathbb{C}} \to R \text{ is nonsingular at } z \in \widehat{\mathbb{C}}\}.$$

Here $\widehat{\mathbb{C}}$ is the Riemann sphere, which is identified with $\mathbb{CP}^1$, the complex projective line, by $[z_1 : z_2] \mapsto z_1/z_2$. (Note that Möbius transformations of the sphere correspond to projective linear transformations.) The group $PSL_2(\mathbb{C})$ of Möbius transformations of $\widehat{\mathbb{C}}$ acts on $\mathcal{P}(R)$ on the right by

$$(j^2_z\psi)g = j^2_{g^{-1}(z)}(\psi \circ g).$$

The action is free with quotient R (see below), with projection given by $j^1_x\psi \mapsto \psi(x)$. Then $\mathcal{P}(R)$ is a principal bundle over R which we propose to call the projective bundle of bases of R. To establish the claims about the action, we need the following lemma.

Lemma 1 *Suppose $U \subset \mathbb{C}$ and $f : U \to \widehat{\mathbb{C}}$ is nonsingular at z_0. Then there is a unique Möbius transformation M such that $j^2_{z_0}f = j^2_{z_0}M$.*

One way to prove this is to first check by direct calculation that for any Möbius transformation M and any $z \in \mathbb{C}$,

$$M(\infty) = M(z) - 2\frac{[M'(z)]^2}{M''(z)},$$

with $M(\infty) = \infty$ if $M''(z) = 0$. (It is left as an exercise to discover the geomtrical content of this equation.) Also by direct calculation, if

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + o(z - z_0)^2$$

then

$$f^{-1}(w) = z_0 + \frac{1}{a_1}(w - a_0) - \frac{a_2}{a_1^3}(w - a_0)^2 + o(w - a_0)^2.$$

We can now read off $M(z_0)$, $M(\infty)$, and $M^{-1}(\infty)$. This information completely determines M, and in fact

$$M(z) = \frac{(a_1^2 - a_0 a_2)(z - z_0) + a_0 a_1}{-a_2(z - z_0) + a_1}$$

has the desired 2-jet. The remaining case occurs when

$$f(z) = \frac{b_0}{z - z_0} + b_1 + o(z - z_0)$$

with $b_0 \neq 0$ and $M(z) = b_1 + b_0/(z - z_0)$ has the desired properties.

In direct analogy with the affine case, a *generalized* projective connection is a connection in $\mathcal{P}(R)$, i.e. an $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form ϕ on $\mathcal{P}(R)$ such that

- $X = \phi(\bar{X})$ for any $X \in \mathfrak{sl}_2(\mathbb{C})$, and
- $\phi \circ TR_g = \mathrm{Ad} g^{-1} \circ \phi$ for any $g \in PSL_2(\mathbb{C})$.

Here $\bar{X}$ denotes the vertical vector field corresponding to X under the representation of $\mathfrak{sl}_2(\mathbb{C})$ into vector fields on $\mathcal{P}(R)$ induced by the action of $PSL_2(\mathbb{C})$, TR_g is the derivative of right multiplication by g on $\mathcal{P}(R)$, and Ad is the adjoint.

Let $\{U_\alpha\}$ be an open cover of R with charts $\{z_\alpha : U_\alpha \to \widehat{\mathbb{C}}\}$. Then we can define sections $\mathcal{X}_\alpha : U_\alpha \to \mathcal{P}(R)$ in the obvious way, namely $\mathcal{X}_\alpha(m) = j^2_{z_\alpha(m)} z_\alpha^{-1}$. Then ϕ is determined by the properties listed above and the pulled down forms $\mathcal{X}_\alpha^* \phi$.

Let the transition functions be denoted $f_{\alpha\beta}$, so that $z_\alpha = f_{\alpha\beta}(z_\beta)$ holds on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Further, define $M_{\alpha\beta} : z_\beta(U_\beta) \to PSL_2(\mathbb{C})$ by $j_z^2 M_{\alpha\beta}(z) = j_z^2 f_{\alpha\beta}$. So $M_{\alpha\beta}(z)$ is the osculating Möbius transformation to $f_{\alpha\beta}$ at z. Exactly as before, we have

$$\mathcal{X}_\beta^* \phi = dM_{\alpha\beta}(z_\beta) + \mathrm{Ad} M_{\alpha\beta}(z_\beta)^{-1} \circ \mathcal{X}_\alpha^* \phi$$

where $dM_{\alpha\beta}(z_\beta)$ is the derivative of $M_{\alpha\beta} \circ z_\beta$ viewed as an $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form. Conversely, a collection $\{\phi_\alpha\}$ of $\mathfrak{sl}_2(\mathbb{C})$-valued 1-forms with ϕ_α defined on $z_\alpha(U_\alpha)$, satisfying

$$\phi_\beta \circ T z_\beta = dM_{\alpha\beta}(z_\beta) + \phi_\alpha \circ T z_\alpha \qquad (2)$$

determines a generalized projective connection on R.

3.2 Projective Connections

Since $PSL_2(\mathbb{C})$ acts effectively on $\widehat{\mathbb{C}}$, the induced representation of $\mathfrak{sl}_2(\mathbb{C})$ into the algebra of vector fields on $\widehat{\mathbb{C}}$ is faithful. Thus we can think of an element $X \in \mathfrak{sl}_2(\mathbb{C})$ as the corresponding vector field. The one (complex) parameter family of Möbius transformations $\exp(tX)$ describes the "time" t flow along X, and X can be profitably thought of as an "infinitesimal Möbius transformation." Quite explicitly, the image of $\mathfrak{sl}_2(\mathbb{C})$ is the set of vector fields of the form $(az^2 + bz + c)\frac{\partial}{\partial z}$; details are left to the reader.

Suppressing the subscripts from the previous section, suppose that f is a biholomorphism from a open subset U of $\widehat{\mathbb{C}}$ to another subset of $\widehat{\mathbb{C}}$ and $M : U \to PSL_2(\mathbb{C})$ is the family of osculating Möbius transformations. Then dM

is an $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form on U, but more can be said. In fact, if v_p is a vector at $p \in U$, then $dM(v_p)$ is an infinitesimal Möbius transformation *vanishing to second order at p.* To see this, note that for $z, q \in U$, in some coordinate near q we have $M(q)(z) = f(z) + o(z-q)^2$ from the definition of M. In particular,

$$M(p+tv_p)(p) = f(p) + o(t)^2 = M(p)(p) + o(t)^2.$$

Then $M(p)^{-1} \circ M(p+tv_p)(p) = p + o(t)^2$, and this is precisely the claim.

As before, we use this information to describe the property which distinguishes projective connections from generalized projective connections. First we define a subbundle $B \subset R \times \mathfrak{sl}_2(\mathbb{C})$ as follows. Let B_α be the (trivial) bundle over U_α whose fiber over m is the subspace of $\mathfrak{sl}_2(\mathbb{C})$ which vanishes to second order at $z_\alpha(m)$. Note that this subspace is one dimensional. Then

$$B = \bigcup_\alpha B_\alpha \Big/ \sim$$

where $(m, X) \in B_\alpha \sim (m, Y) \in B_\beta$ if $X = (M_{\alpha\beta}(q))_* Y$.

We have shown that $dM_{\alpha\beta}(z_\beta)$ is a section of B over $U_{\alpha\beta}$. With this in mind, we define a projective connection to be a generalized projective connection ϕ such that every $\mathcal{X}_\alpha^* \phi$ takes values in B.

This line bundle B is actually isomorphic to the cotangent bundle T^*R. There are cleaner ways to establish this than the one which follows, but the author found this direct approach most satisfying. The reader is invited to exhibit the dual pairing of B_m with $T_m R$.

Lemma 2 *If M is a Möbius transformation, $p = M(q)$, and neither p nor q is ∞, then*

$$(z-q)^2 M'(z) = \frac{1}{M'(q)} (M(z) - p)^2.$$

This is because

$$\frac{(M(z)-p)^2}{M'(z)(z-q)^2}$$

is rational with no zero or pole in $\mathbb{C}$ and limit $M'(q)$ as $z \to q$.

Theorem 3 *B and T^*R have the same transition functions.*

Suppose ω is a (meromorphic) section of B. In the coordinate z_β, we write $\omega = \omega(z_\beta)(z - z_\beta)^2 \frac{\partial}{\partial z}$. Using $w = M_{\alpha\beta}(z_\beta)(z)$, we compute

$$\begin{array}{rcll} \omega(z_\alpha)(w-z_\alpha)^2 \frac{\partial}{\partial w} & = & (M_{\alpha\beta}(z_\beta))_* \left(\omega(z_\beta)(z-z_\beta)^2 \frac{\partial}{\partial z}\right) & \text{by definition of } B \\ & = & \omega(z_\beta)(z-z_\beta)^2 (M_{\alpha\beta}(z_\beta))'(z) \frac{\partial}{\partial w} & \\ & = & \omega(z_\beta) \frac{1}{(M_{\alpha\beta}(z_\beta))'(z_\beta)} (w - z_\alpha)^2 \frac{\partial}{\partial w} & \text{by lemma} \\ & = & \omega(z_\beta) \frac{1}{f'_{\alpha\beta}(z_\beta)} (w-z_\alpha)^2 \frac{\partial}{\partial w} & \text{by definition of } M_{\alpha\beta} \end{array}$$

So $\omega(z_\alpha) f'_{\alpha\beta}(z_\beta) = \omega(z_\beta)$, which says that $\omega(z_\alpha) dz_\alpha = \omega(z_\beta) dz_\beta$, establishing the theorem.

Now we find that the 1-cocycle $M = \{dM_{\alpha\beta}(z_\beta)\}$ represents a class in $H^1(R, T^*R \otimes T^*R)$ which is canonically associated with R. The cocycle condition

$$0 = (\delta M)_{\alpha\beta\gamma} = dM_{\alpha\beta}(z_\beta) + dM_{\beta\gamma}(z_\gamma) + dM_{\gamma\alpha}(z_\alpha)$$

is exactly the infinitesimal form of the more obvious identity

$$\mathrm{id} = M_{\alpha\beta}(z_\beta) \circ M_{\beta\gamma}(z_\gamma) \circ M_{\gamma\alpha}(z_\alpha).$$

The fact that the cohomology class of M is independent of the coordinate covering $\{U_\alpha, z_\alpha\}$ is left to the reader.

In view of equation 2 of the previous section, a projective connection is a 0-cochain $\phi \in C^0(R, T^*R \otimes T^*R)$ such that $\delta\phi = M$. This is essentially the definition of projective connection which appears in [G]. Note that the existence of a projective connection is equivalent to the vanishing of M in cohomology. Using the Riemann-Roch theorem, it can be shown (see [G] again) that the projective structures on a compact Riemann surface of genus $g > 1$ form a complex affine space of dimension $3g - 3$.

3.3 The Developing Map and Holonomy

Since a Riemann surface is after all a 1-dimensional manifold, any projective connection on R is necessarily flat. In this section we hope to show that a projective connection on R is equivalent to a projective *structure* on R. By a projective structure, we mean an atlas on R compatible with the given one whose transition functions $f_{\alpha\beta}$ are Möbius transformations. (The disclaimer "hope" is included because the author feels compelled to confess a certain nagging fear that what follows is irretrievably wrong.)

Since $PSL_2(\mathbb{C})$ acts on the left on $\widehat{\mathbb{C}}$ by Möbius transformations, there is a bundle over R with fiber $\widehat{\mathbb{C}}$ associated to $\mathcal{P}(R)$. What is this bundle? We will write it $\widehat{T}R$. Intuitively, it can be thought of as being obtained from TR, the tangent bundle, by adding a point at infinity to each tangent space. Precisely, let us agree to distinguish the point ∞ on the abstract fiber. Since $PSL_2(\mathbb{C})$ acts by automorphisms of $\widehat{\mathbb{C}}$, we obtain an equivalent bundle (ahem) by first reducing the principal bundle $\mathcal{P}(R)$ to a principal bundle with group stabilizing ∞. The stabilizer of ∞ is $A(1)$, the group of affine isomorphisms of $\mathbb{C}$, and it's not hard to explicitly obtain a reduced bundle isomorphic to $\mathcal{A}(\mathcal{R})$, the *affine* bundle of bases.

Distinguishing the point ∞ on the abstract fiber $\widehat{\mathbb{C}}$ gives a distinguished point in each fiber $\infty \in \widehat{T}_m R$. This seems like a very unnatural thing to do, and it is. In fact, it is as unnatural as distinguishing a point in an affine space and calling it the origin. This latter unnatural act, it should be noted, is exactly what gave us the developing map in the affine case.

Suppose that ϕ is a projective connection on R. Choose a point $m \in R$ and think of $\widetilde{R}$, the universal cover, as the space of homotopy classes of paths with initial point m. We define the developing map $\mathrm{dev} : \widetilde{R} \to \widehat{T}_m R$ by letting $\mathrm{dev}(\gamma)$ be the image of $\infty \in \widehat{T}_{\gamma(1)}R$ under parallel translation along γ to $\widehat{T}_m R$. Because ϕ is flat, $\mathrm{dev}(\gamma)$ depends only on the homotopy class of γ rel endpoints. The argument from the affine case carries through to yield the fact that dev is defined up to deck transformation of $\widetilde{R}$ and projective isomorphism of the range, and the holonomy $h : \Pi_1(R) \to PSL_2(\mathbb{C})$ satisfies

$$h(\alpha) \circ \mathrm{dev} = \mathrm{dev} \circ \alpha.$$

Once it is established that dev is locally injective, it follows that the developing map and holonomy together determine a projective structure on R. It is again left to the reader to check that a projective structure determines a projective connection.

The argument that dev is locally injective is essentially identical to that of the affine case, or at least it should be. A chart z near m should establish an identification of $\widehat{T}_pR$ with $\widehat{\mathbb{C}}$ with the distinguished point $\infty \in \widehat{T}_pR$ corresponding to $z(p)$. Then parallel translation along γ with initial point m to $\gamma(t)$ is effected by a Möbius transformation M_t. The definition of projective connection guarantees that $M_t \circ z(m) = z(m) + o(t)^2$. So the distinguished point on a fiber does *not* go to the distinguished point on a nearby fiber for small t, and this is again the decisive case.

Question 1 *How exactly does z identify $\widehat{\mathbb{C}}$ with $\widehat{T}_pR$? What is the definition of $\mathcal{P}(R)$ analogous to the definition of $\mathcal{A}(\mathcal{M})$ as pairs (b,t)?*

3.4 An Open Question

It is known ([G] again) that any representation $h : \pi_1(R) \to PSL_2(\mathbb{C})$ of the fundamental group of a closed (topological) surface which arises as the holonomy of a projective structure actually lifts to a representation into $SL_2(\mathbb{C})$. It is also known that there can be no invariant point or geodesic in $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$ under the action of the image group, i.e. the representation h must be *nonelementary.* The conjecture (due to Gunning, but a different paper) is that any representation which satisfies these two necessary conditions can in fact be realized as the holonomy of some projective structure on some Riemann surface. Proofs have been announced twice recently ([Ga], [K]), but the question is in fact still open.

4 The Schwarzian Derivative

Closely related to this discussion is the classical Schwarzian derivative. Most complex analysis texts which mention the Schwarzian simply define it as a fairly unpleasant looking differential operator and proceed to derive some consequences. Some references, especially [T2] and [H], give more intrinsic discussions which this author, for one, was unable to make sense of without a great deal of effort. Both of these papers provide recipes (subtly distinct from each other) for deducing the classical formula, with neither of which has this author had any success whatsoever. Here is another version, worked out in more or less gory detail.

Suppose $U \subset \widehat{\mathbb{C}}$ is open and $f : U \to \widehat{\mathbb{C}}$ is holomorphic and locally injective, and let $M : U \to PSL_2(\mathbb{C})$ be the osculating Möbius transformation. Then the Schwarzian derivative of f is $Sf = dM$. As discussed earlier, Sf is a section of $T^*U \otimes T^*U$, i.e. a quadratic differential on U. We now derive an explicit formula for Sf in terms of f and its derivatives.

Lemma 4 *If*

$$N_t(x) = \frac{a(t)(x-t)+b(t)}{c(t)(x-t)+d(t)}$$

with $N_z = id$, then

$$\left.\frac{\partial}{\partial t}\right|_z N_t(x) = -c'(z)x(x-z) - d'(z)x + a'(z)(x-z) + b'(z) - 1.$$

The calculation is straightforward.

Lemma 5 *If M_t is the osculating Möbius transformation to f at t, and $M_z = id$, then*

$$\left.\frac{\partial}{\partial t}\right|_z M_t(x) = \frac{1}{2} f'''(z)$$

Refer to lemma 1 and the explicit determination of M in terms of f and use the previous lemma.

Lemma 6 $dM = \frac{1}{2}(M_z^{-1} \circ f)'''(z)(x-z)^2 \frac{\partial}{\partial x} \otimes dz$

We know that dM is a 1-form, so we only need to evaluate dM on a vector $\frac{\partial}{\partial z}$ at z. The result will be a vector field on $\widehat{\mathbb{C}}$ which we previously showed will be a constant times $(x-z)^2 \frac{\partial}{\partial x}$. We just need that constant.

$$dM(\left.\frac{\partial}{\partial z}\right|_z) = (M_z^{-1})_* \left.\frac{\partial}{\partial t}\right|_z M_t = \left.\frac{\partial}{\partial t}\right|_z (M_z^{-1} \circ M_t)$$

The previous lemma applies to $M_z^{-1} \circ M_t$ and $M_p^{-1} \circ f$. Remembering how $\mathfrak{sl}_2(\mathbb{C})$ is represented on vector fields, this shows the constant is $\frac{1}{2}(M_z^{-1} \circ f)'''(z)$, and the lemma follows.

Lemma 7

$$(M_z^{-1} \circ f)'''(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

If

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + o(z-z_0)^3$$

then we calculated long ago that

$$M_{z_0}^{-1}(w) = z_0 + \frac{1}{a_1}(w-a_0) - \frac{a_2}{a_1^3}(w-a_0)^2 + o(w-a_0)^2$$

but now we need the coefficient on the cubic term. In general, the easiest way to obtain this coefficient is to let $N(w) = \frac{a(w-p)+b}{c(w-p)+d}$ and examine the quadratic and cubic terms of

$$a(w-p) + b = [c(w-p)+d](\text{Taylor series for } N \text{ about } p).$$

For the record, the number is a_2^2/a_1^5. Composing the power series, one finds, finally, that

$$M_{z_0}^{-1} \circ f(z) = z + (\frac{a_3}{a_1} - \frac{a_2^2}{a_1^2})(z-z_0)^3 + o(z-z_0)^3$$

This establishes the lemma and the right of the author to refuse to repeat any of these calculations ever again.

Choosing the isomorphism $dz^2 \mapsto \frac{1}{2}(x-z)^2 \frac{\partial}{\partial x} \otimes dz$ and combining the results above, we obtain the usual expression for the Schwarzian, namely

$$Sf(z) = \left(\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right) dz^2.$$

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