A Brief History of Morse Homology

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Abstract

Morse theory was originally due to $Marston\ Morse$ [5]. It gives us a method to study the topology of a manifold using the information of the critical points of a Morse function defined on the manifold. Based on the same idea, Morse homology was introduced by $Thom$, $Smale$, $Milnor$, and $Witten$ in various forms. This paper is a survey of some work in this direction. The first part of the paper focuses on the classical flow line approach by $Thom$, $Smale$, and $Milnor$. The second part of the paper will concentrate on $Witten$'s alternative and powerful approach using Hodge theory.

1 Basic concepts in classical Morse theory

In this paper, let M be a compact n dimensional manifold and $f : M \to \mathbb{R}$ be a smooth function. A point $p \in M$ is called a **critical point** if the induced map $f_* : TM_p \to T\mathbb{R}_{f(p)}$ has rank zero. In other words, p is a critical point of f if and only if in any local coordinate system around p one has

$$\frac{\partial f}{\partial x_1}(p) = \frac{\partial f}{\partial x_2}(p) = \cdots = \frac{\partial f}{\partial x_n}(p) = 0.$$

The real value $f(p)$ is then called a **critical value** of f. A critical point p is said to be **non-degenerate** if, in a local coordinate system around p, the Hessian $(\frac{\partial^2 f}{\partial x^i \partial x^j}(p))$ of f at p is non-degenerate. For a non-degenerate critical point, the number of negative eigenvalues of the Hessian is its **Morse index**. If all critical points of f are non-degenerate, the function f is then called a **Morse function**. There is a theorem which says that for any closed smooth manifold M, a generic C^k function $f : M \to \mathbb{R}$ is Morse.

The local form of a Morse function is nicely described in the following lemma [3]:

Theorem 1.1 (Morse lemma) Let p be a non-degenerate critical point of f. Then there is a local coordinate system $(y^1, ..., y^n)$ in a neighborhood U of p with $y^i(p) = 0$ for all i and such that the identity:

$$f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$$

holds throughout U, where λ is the Morse index of f at p.

An easy corollary of Morse lemma is that non-degenerate critical points are isolated.

The classical idea of Morse theory is to study the submanifold $M^a = \{p \in M | f(p) \leq a\}$, where a is not a critical value of f. Let's assume that all M^a's are compact. There are two important theorems concerning the change of homotopy type of M^a as a increasing [3]:

Theorem 1.2 If there is no critical value of f within the interval $[a, b]$, then M^a is diffeomorphic to M^b. Moreover, M^a is a deformation retract of M^b.

Theorem 1.3 If there is only one non-degenerate critical point within the interval $[a, b]$ of index λ, then the homotopy type of M^b is obtained from that of M^a with a λ-cell attached.

These theorems give rise to the following famous **strong Morse inequalities**, which are lower bounds on the number of critical points of f. They are bridges between analysis and topology [6].

Theorem 1.4 (**Morse inequalities**) Let b_k denote the k^{th} Betti number of M, i.e. the dimension of the cohomology group $H^k(M, \mathbb{R})$, and let c_k denote the number of index k critical points of f. If all critical points of f are non-degenerate, then

$$c_k - c_{k-1} + c_{k-2} - \cdots + (-1)^k c_0 \geq b_k - b_{k-1} + b_{k-2} - \cdots + (-1)^k b_0 \quad \text{for each } k,$$

$$\sum_{k=0}^{n}(-1)^k c_k = \sum_{k=0}^{n}(-1)^k b_k = \chi(M).$$

$Poincaré$ asked a classical question about this inequality, the famous **Poincaré conjecture**: If an n-manifold M has the same Betti numbers as the n-sphere, does it follow that it can have a Morse function with exactly two critical points and hence be homeomorphic to S^n? This was proved to be true for $n = 4$ by $Freedman$ [2], and for $n \geq 5$ by $Smale$ [9]. [4] gives a Morse-theoretical proof of $Smale$'s famous **h-cobordism theorem** which plays an essential role in $Smale$'s proof of the $Poincaré$ conjecture in dimension greater than 5.

Theorem 1.5 ($Smale$'s h-cobordism theorem) Let W^{n+1} be a compact manifold with boundary. If $\partial W = V_1^n \cup V_2^n$ such that each V_i is simply connected and is a deformation retract of W, then W is diffeomorphic to $V_i \times [0,1]$, and so V_1 is diffeomorphic to V_2.

If we take a closer look at the procedure of attaching cells, we will get the following handle decomposition of the manifold M:

Theorem 1.6 Every connected closed smooth manifold M^n is diffeomorphic to a union of finitely many handles $H_\lambda^n = B^\lambda \times B^{n-\lambda}$ (B^λ is a λ-dim ball), where the handles H_λ^n are in one-to-one correspondence with the critical points of index λ. Conversely, given a decomposition of the manifold into a sum of handles, there exists a Morse function which gives induces the same decomposition.

But even if two manifolds have the same number of cells in each dimension, they are not necessarily homotopy equivalent to each other. In order to know the homotopy type of the manifold, we need to study the attaching maps. The first successful attempts were made by $Thom$ [10], $Smale$ [8], and $Milnor$ [3] (40's - 60's). The idea of Morse homology come from them.

2 Classical approach of Morse homology

Now let g be a Riemannian metric on M. Consider the negative gradient flow φ of a Morse function f:

$$\varphi : \mathbb{R} \times M \to M,$$

$$\frac{\partial}{\partial t}\varphi(t,x) = -\nabla f(\varphi(t,x)),\ \ \varphi(0,\cdot) = id_M.$$

If p is a critical point of f, one can define the **stable manifold** and **unstable manifold** as:

$$W_p^s = \{x \in M | \lim_{t\to\infty} \varphi(t,x) = p\},$$

$$W_p^u = \{x \in M | \lim_{t\to-\infty} \varphi(t,x) = p\}.$$

If p is a non-degenerate critical point of f, $T_p W_p^u$ is the negative eigenspace of the Hessian $H(f,p)$, so W_p^u is an embedded open disk with dimension equal to ind(p), the Morse index of p. Similarly W_p^s is an embedded open disk with dimension n-ind(p). $Thom$ first recognized that the decomposition of M into unstable manifolds gives a cell decomposition which is homologically equivalent to the one described in Theorem 1.6. But, in general, this decomposition is not a CW-complex from which one can compute homology group. However, in the 50's, $Smale$ found that if we put an additional requirement on the Riemannian metric, the cells we get will attach to each other properly and give us the desired CW-complex structure. The requirement on the pair (f,g) is called **Morse-Smale condition**: namely, f is a Morse function and for every pair of critical points p and q, the unstable manifold W_p^u of p is transverse to the stable manifold W_q^s of q. Moreover, **Smale** discovered that this transversal requirement holds for a generic Riemannian metric on M.

For any pair of critical points p and q, we define a **flow line** from p to q to be a map $\gamma : \mathbb{R} \to M$ such that $\gamma'(t) = -\nabla f(\gamma(t))$ and $\lim_{t\to-\infty}\gamma(t) = p$, $\lim_{t\to\infty}\gamma(t) = q$. There is a natural $\mathbb{R}$ action on the set of flow lines from p to q by precomposition with translations of $\mathbb{R}$. Let $\mathcal{M}(p,q)$ denote the moduli space of flow lines from p to q, modulo $\mathbb{R}$ action. Then actually we have $\mathcal{M}(p,q) = (W_p^u \cap W_q^s)/\mathbb{R}$. We know dim($W_p^u$)=ind($p$) and dim($W_q^s$)=$n$-ind($q$). Because of the transversality condition, when $p \neq q$, one can easily deduce that:

$$\begin{aligned}\dim(W_p^u \cap W_q^s) &= \dim(W_p^u) + \dim(W_p^u) - n \\ &= \mathrm{ind}(p) + (n - \mathrm{ind}(q)) - n \\ &= \mathrm{ind}(p) - \mathrm{ind}(q)\end{aligned}$$

When $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, the moduli space has dimension zero. We want to count how many flow lines there are from p to q. Before we can proceed, we first need to choose orientations on the moduli space and secondly show that the moduli spaces are compact.

Orientations can be selected as follows [6]. Choose an orientation of the unstable manifold W_p^u for each critical point p. At any point on the flow line, we have a canonical isomorphism:

$$\begin{aligned} TW_p^u &\simeq T(W_p^u \cap W_q^s) \oplus (TM/TW_q^s) \\ &\simeq T_\gamma \mathcal{M}(p,q) \oplus T_\gamma \oplus T_q W_q^u. \end{aligned} \tag{1}$$

The first isomorphism comes from the Morse-Smale transversality condition. The isomorphism $T(W_p^u \cap W_q^s) \simeq T_\gamma \mathcal{M}(p,q) \oplus T_\gamma$ holds because $\dim(W_p^u \cap W_q^s) = \mathrm{ind}(p) - \mathrm{ind}(q)$. By translating the subspace $T_q W_q^u \subset T_q M$ along γ while keeping it complementary to TW_q^s we get the isomorphism $TM/TW_q^s \simeq T_q W_q^u$. The orientation on $\mathcal{M}(p,q)$ is chosen such that isomorphism (1) is orientation-preserving.

For compactness, there is a theorem says that one can get the closure of $\mathcal{M}(p,q)$ by adding some “broken” flow lines, i.e. flow lines that “pass” other critical points before reaching target. And, in fact, any such “broken” flow line lies in the boundary of the moduli space $\mathcal{M}(p,q)$.

Theorem 2.1 If M is closed and (f,g) is Morse-Smale, then for any two critical points p and q, the moduli space $\mathcal{M}(p,q)$ has a natural compactification to a smooth manifold with corners $\overline{\mathcal{M}(p,q)}$ whose codimension k stratum is

$$\overline{\mathcal{M}(p,q)_k} = \bigcup_{r_1,\cdots,r_k \in \mathrm{Crit}(f)} \mathcal{M}(p,r_1) \times \mathcal{M}(r_1,r_2) \times \cdots \times \mathcal{M}(r_{k-1},r_{k-2}) \times \mathcal{M}(r_k,q)$$

where $p, r_1, \cdots, r_k, q$ are distinct. When $k=1$, as oriented manifolds we have

$$\partial \overline{\mathcal{M}(p,q)} = \bigcup_{r \in \mathrm{Crit}(f), r \neq p, r \neq q} (-1)^{\mathrm{ind}(p)+\mathrm{ind}(r)+1} \mathcal{M}(p,r) \times \mathcal{M}(r,q).$$

Where a **smooth manifold with corners** is a second countable Hausdorff space such that each point has a neighborhood with a chosen homeomorphism with $\mathbb{R}^{n-k} \times [0,\infty)^k$ for some $k \leq n$, and the transition maps are smooth.

Finally we can define the **Morse complex**. Let $\mathrm{Crit}_k(f)$ be the set of index k critical points of f. The chain group $C_k(f)$ is a free Abelian group generated by

$\mathrm{Crit}_k(f)$:

$$C_k(f) = \mathbb{Z}\mathrm{Crit}_k(f), \quad k = 0, \cdots, n.$$

And the boundary operator $\partial_k : C_k(f) \to C_{k-1}(f)$ is defined by counting the algebraic number of flow lines connecting p and q:

$$\partial_k p = \sum_{q \in \mathrm{Crit}_{k-1}(f)} \#\mathcal{M}(p,q) \cdot q.$$

The sign of each flow line is decided by comparing the natural orientation on the flow line induced by the $\mathbb{R}$ action with the orientation of $\mathcal{M}(p,q)$.

To get $\partial^2 = 0$ we observe a special case of theorem 2.1. When $\mathrm{ind}(p) = \mathrm{ind}(q) + 2$, we have

$$\begin{aligned} \partial\overline{\mathcal{M}(p,q)} &= \bigcup_{r \in \mathrm{Crit}(f) \quad, r \neq p, r \neq q} (-1)^{\mathrm{ind}(p)+\mathrm{ind}(r)+1} \mathcal{M}(p,r) \times \mathcal{M}(r,q) \\ &= \bigcup_{r \in \mathrm{Crit}_{\mathrm{ind}(p)-1}(f)} \mathcal{M}(p,r) \times \mathcal{M}(r,q). \end{aligned}$$

The above union is over index $\mathrm{ind}(p) - 1$ critical points only because otherwise $\mathcal{M}(p,r) = \emptyset$ or $\mathcal{M}(r,q) = \emptyset$. Now if $p \in \mathrm{Crit}_k(f)$ and $q \in \mathrm{Crit}_{k-2}(f)$, then $\partial^2 p = \partial_{k-1}\partial_k p$ counts “2-broken” flow lines connecting p and q. These flowlines are on the boundary of the 1-dimensional moduli space $\mathcal{M}(p,q)$. Since the algebraic sum of the boundary points of a compact oriented 1-manifold is zero, we get the identity $\partial^2 = 0$. More precisely, we have [6]:

$$\begin{aligned} \langle \partial^2 p, q \rangle &= \sum_{r \in \mathrm{Crit}_{k-1}(f)} \langle \partial_k p, r \rangle \langle \partial_{k-1} r, q \rangle \\ &= \# \bigcup_{r \in \mathrm{Crit}_{k-1}(f)} \mathcal{M}(p,r) \times \mathcal{M}(r,q) \\ &= \#\partial\overline{\mathcal{M}(p,q)} = 0. \end{aligned}$$

The **Morse homology** is defined to be the homology of this chain complex $(C_*(f), \partial)$. This algebraic formulation is due to $Thom$, $Smale$, and $Milnor$. The most fundamental theorem in Morse theory is that the Morse homology defined above is canonically isomorphic to the singular homology of the underlying compact manifold. This immediately verifies the Morse inequality (Theorem 1.4).

3 Witten's alternative approach

In his wonderful paper [11], $Witten$ rediscovered the way of computing the cohomology group of a manifold in terms of the critical points of a Morse function. His approach is along different lines with the classical approach by $Thom$, $Smale$, and $Milnor$. Before giving $Witten$'s idea, let's recall a little bit of Hodge theory.

Let M be an n-dimensional oriented smooth manifold. For all $0 \leq k \leq n$, we have $\Lambda^k T_p^* M \simeq \Lambda^{n-k} T_p^* M$ because as vector spaces they have the same dimension. An explicit isomorphism is given as follows. There is a natural isomorphism $TM \simeq T^*M$ induced by the Riemannian metric g. So the inner product in TM will induce an inner product in T^*M. Let $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ be a positively oriented orthonormal basis for T_p^*M. Then we get a natural linear isomorphism:

$$* : \Lambda^k T_p^* M \to \Lambda^{n-k} T_p^* M$$

by setting

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n.$$

This operator is called the **Hodge star**. Varying $p \in M$ gives a linear isomorphism

$$* : \Omega^k(M) \to \Omega^{n-k}(M).$$

where $\Omega^k(M)$ (resp. $\Omega^{n-k}(M)$) is the vector space of all k-forms (resp. $(n-k)$-forms) on M.

We define a linear operator

$$\delta = (-1)^k *^{-1} d* = (-1)^{n(k+1)+1} * d*$$

by requiring the following diagram to be commutative:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{*} & \Omega^{n-k}(M) \\ \delta \downarrow & & \downarrow d \\ \Omega^{k-1}(M) & \xrightarrow[(-1)^k *]{} & \Omega^{n-k+1}(M) \end{array}$$

Now the operator Δ defined by

$$\Delta = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M)$$

is called the **Laplacian**. A form $\omega \in \Omega^*(M)$ such that $\Delta\omega = 0$ is called a **harmonic form**. A necessary and sufficient condition for ω to be harmonic is that $d\omega = 0$ and $\delta\omega = 0$. This in particular means that every harmonic form is closed.

Denote by $\mathbf{H}^k(M)$ the set of all harmonic k-forms on M, i.e.

$$\mathbf{H}^k(M) = \{\omega \in \Omega^k(M) | \Delta\omega = 0\}$$

Since every harmonic form is closed, we get a linear map

$$\mathbf{H}^k(M) \to \mathrm{H}^k_{DR}(M)$$

by taking the de Rham cohomology class. The De Rham theorem says that $\mathrm{H}^k_{DR}(M)$ is isomorphic to $\mathrm{H}^k(M, \mathbb{R})$. The Hodge theorem tells us that the linear map from H^k to H^k_{DR} is actually an isomorphism.

Theorem3.1 (**Hodge theorem**) An arbitrary de Rham cohomology class of an oriented compact Riemannian manifold can be represented by a unique harmonic form. In other words, the natural map $\mathbf{H}^k(M) \to \mathrm{H}^k_{DR}(M)$ is an isomorphism.

Now let's start from the manifold M, Morse function f, and Riemannian metric g as before. Consider the de Rham complex of M:

$$\Omega^0 \to \Omega^1 \to \cdots \to \Omega^n.$$

We decompose Ω^k into the direct sum of finite-dimensional eigenspaces according to Laplacian Δ:

$$\Omega^k = \bigoplus_{\lambda} \Omega^k_\lambda,$$

where $\Omega_\lambda^k = \{\omega \in \Omega^k | \Delta\omega = \lambda\omega\}$. The Hodge theory implies that

$$\Omega_0^k \simeq \mathbf{H}^k \simeq \mathrm{H}_{DR}^k(M) \simeq \mathrm{H}^k(M, \mathbb{R}).$$

From the definition of Δ we see that

$$\Delta d = d\delta d + \delta d^2 = d\delta d + d^2\delta = d\Delta.$$

So we can restrict d to $\Omega_\lambda^* = \bigoplus_{k=1}^n \Omega_\lambda^k$,denoted by d_λ, and get the following exact sequence:

$$0 \longrightarrow \Omega_\lambda^0 \xrightarrow{d_\lambda} \Omega_\lambda^1 \xrightarrow{d_\lambda} \cdots \xrightarrow{d_\lambda} \Omega_\lambda^n \longrightarrow 0$$

The reason of exactness is as follows. For any $\omega \in \Omega_\lambda^k$, if ω is in the kernel of d_λ, then $d_\lambda \omega = 0$, and we have

$$\omega = \frac{1}{\lambda}\lambda\omega = \frac{1}{\lambda}\Delta\omega = \frac{1}{\lambda}(d\delta + \delta d)\omega = \frac{1}{\lambda}d\delta\omega = d(\frac{1}{\lambda}\delta\omega).$$

Since Δ commutes with both d and δ, we get $d(\frac{1}{\lambda}\delta\omega) \in \Omega_\lambda^{k-1}$, which means the above sequence is exact. Based on the above facts, one can easily conclude that the complexes $\Omega_a^* = \bigoplus_{\lambda \le a} \Omega_\lambda^*$ have $\mathbf{H}^*(M)$ as their cohomology for any $a > 0$.

Here comes $Witten$'s idea [1]. Conjugating d by multiplication with e^{sf}, $s \in \mathbb{R}$, gives an operator $d_s = e^{-sf} \circ d \circ e^{sf}$ which satisfies $(d_s)^2 = 0$. Because all we do is conjugation, it's easy to see that this co-boundary operator yields a co-homology group $\mathbf{H}_s^*(M)$ which is again isomorphic.

$$\mathbf{H}_s^*(M) = \ker(d_s)/\mathrm{Im}(d_s) \simeq \mathrm{H}_{DR}^*(M) \simeq \mathrm{H}^*(M, \mathbb{R})$$

We can also compute $\mathbf{H}_s^*(M)$ using Hodge theory. This time let's consider the operator $\Delta_s = d_s\delta_s + \delta_s d_s$ and the decomposition $\Omega^*(s) = \bigoplus_\lambda \Omega_\lambda^*(s)$, where $\Omega_\lambda^*(s)$ is just the eigenspace of Δ_s as before. We also define complexes $\Omega_a^*(s) = \bigoplus_{\lambda \le a} \Omega_\lambda^*(s)$ spanned by all eigenforms of Δ_s with eigenvalues $\lambda \le a$.

This curve of chain complex $\Omega_a^*(s)$ is the essence of $Witten$'s version of Morse homology. He stated that if s is large enough, the dimension of this chain complexes will be independent of s, so denoted by $\Omega_a^*(\infty)$. And also one has:

1. dim($\Omega_a^k(\infty)$)= # of critical points of index k

2. the boundary operator induced by d on $\Omega_a^k(\infty)$ is carried by the connecting orbits from the critical points of index k to those of index $k+1$.

This gives us a direct link between the homology of the underlying manifold and Hodge theory. See [1] for an intuitive idea of why the above statements might be true. However, $Witten$ did not prove, using strict mathematical arguments, the above assertion in his great paper [11]. For a complete proof of $Witten$'s idea, see Helffer and Sjöstrand [7].

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