THE GAUSS-BONNET THEOREM AND ITS APPLICATIONS

CHENCHANG ZHU

ABSTRACT. In this paper we survey some developments and new results on the proof and applications of the Gauss-Bonnet theorem. Our special emphasis is the relation of this theorem to different areas including characteristic classes, probability, polyhedra and physics.

1. INTRODUCTION

The Gauss-Bonnet theorem is an important theorem in differential geometry. It is intrinsically beautiful because it relates the curvature of a manifold—a geometrical object—with the its Euler Characteristic—a topological one. In this article, we shall explain the developments of the Gauss-Bonnet theorem in the last 60 years.

First of all, we will state this theorem in its initial form and find out that, even in this case, there are several interesting applications of it in the area of low dimensional manifolds.

Then, we'll generalize this initial form in two directions and exhibit the respective results. It turns out that the second one is more important than the first one. It aroused the interest of mathematicians about 50 years ago and gives great insight into several theories. We will outline one of the proofs for the theorem in this general version. The proof uses the characteristic classes of vector bundles on the manifolds, which will give us a better understanding of the theorem itself.

Moreover, we will discuss other efforts to prove this theorem. By looking at its polyhedral version, we can see that it is dual to the Gram-Sommerville formula. On the other hand, from the viewpoint of probability, it relates to a heat kernel and can be proved by an analytical method.

Finally, we are going to look at its application in physics and describe the derivation of the optical Berry phase from our Gauss-Bonnet theorem.

2. BASIC DEFINITIONS FOR THE THEOREM

1. Gauss Map: Let M be an oriented hypersurface in $n+1$ dimensional Euclidean space. Then M has trivial normal bundle by the fact that M is oriented. Hence, at every point $x \in M$, there is a normal vector n_x pointing outside in $\mathbb{R}^{n+1}$, which has norm 1 and changes smoothly w.r.t. x. Define the Gauss map γ from M to S^n by $\gamma(x) = n_x$. (Here we actually take $T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$ as the same Euclidean space for every x in M.)

2. Gauss-Kronecker Curvature: (We will also call it Gauss curvature in this paper) Let N, M be Riemannian manifolds with Levi-Civita Connection $\bigtriangledown^N,\ \bigtriangledown^M$ and M be a submanifold of N. For $n \in T_xM^{\perp}, X, Y \in T_xM, \forall x \in M,$ we define the second fundamental form of M w.r.t. N by $l_n(X,Y) = -< n, \bigtriangledown^N_X Y >$. It is linear and symmetric in X and Y. (see [J] 3.4). Thus $\exists S_nX \in T_xM$ such that $l_n(X,Y) =< S_nX, Y >$. For a unit normal vector field n, $S_n : T_xM \longrightarrow T_xM$ then is a self adjoint linear map w.r.t. the metric $<,>$. Finally, the Gauss-Kronecker curvature of M in the direction n is defined by

$$K_n = \det S_n.$$

It is not hard to prove that K_n is also $\det(d\gamma)$, where γ is the Gauss map and n is its corresponding normal vector field in its definition.

3. Euler Characteristic:

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(a) Topological definition: The Euler characteristic of a manifold M^m is defined as,

$$\chi(M^m) := \sum_{i=0}^{m} (-1)^i \beta_i,$$

where $\beta_i = \mathrm{rank}(H^i(M))$.

(b) Geometrical definition: Let M, N be oriented smooth manifolds, S be an oriented submanifold of N, and $\dim M + \dim S = \dim N$. Let $f \in C^\infty(M, N)$, $p \in f^{-1}(S)$, and f intersects S transversally at p. Let $\pi : \mathbb{R}^n \longrightarrow 0 \times \mathbb{R}^{n-s}$ be the projection and ψ be the chart near p, such that $d\psi(T_pS) = \mathbb{R}^s \times 0$, then $\pi \circ \psi \circ f$ is an isomorphism near p. Hence we can ask $Sgn(\ \pi \circ \psi \circ f\ )$ to be the sign of the determinant of $T_p(\pi \circ \psi \circ f)$. Now define the intersection number of f and S as:

$$\mathrm{I}(f, S) := \Sigma_{p \in f^{-1}(S)} \mathrm{Sgn}(f, S)_p.$$

There is an important property of the intersection number that if f and g is homotopic, then $\mathrm{I}(f, S) = \mathrm{I}(g, S)$. We also know that every map can be homotopic to a map transversing to S provided some conditions of dimensions. So $\mathrm{I}(f, S)$ is meaningful even if f doesn't transverse to S.

Now if M is also a submanifold of N, we can define the intersection number of M and S as $\mathrm{I}(M, S) := \mathrm{I}(i, S)$, where i is the inclusion map. Finally, let E_0 be the 0-section of TM; the Euler characteristic of M is defined by:

$$\chi(M) := \mathrm{I}(E_0, E_0).$$

In fact, these two definitions are equivalent.

3. The Gauss-Bonnet Theorem for Codimension 1

After we defined the Gauss map, Gauss curvature and Euler characteristic, we can describe the Gauss-Bonnet theorem without any difficulty.

Theorem 3.1. *(original Gauss-Bonnet theorem) Let M be an even dimensional compact smooth hypersurface in the Euclidean space, then*

$$v(m)^{-1} \int_M K_n(x) d\mu_M = (1/2)\chi(M) = deg(\gamma),$$

where m is the dimension of M, $v(m)$ is the volume of S^m, and n is the normal vector field appeared in the definition of Gauss map γ.

Even in its original form, the Gauss-Bonnet theorem is rather useful for low dimensional manifolds. Because it links the curvature and the Euler characteristic, we can always predict something topological for a manifold with constant-sign curvatures. We cite two examples here:

Example 3.2. For a 2-dimensional manifold M, $\chi(M) = 2 - 2g$. If M has positive curvature K, then $2\pi \cdot \chi(M) = \int K d\mu > 0$, hence, topologically, M can only be S^2. Moreover $\gamma(M) \subset S^2$ is closed (by compactness of M) and open (by $\det(d\gamma) = K > 0$), so $\gamma(M) = S^2$. Therefore, $\gamma : M \longrightarrow S^2$ is onto. Furthermore, we can use the Gauss-Bonnet theorem to prove that γ is indeed injective. In fact in [S]-Chap 6, there is even a theorem for more curious readers: (we write it as a statement)

Statement Let M be a compact connected 2-manifold, and $f : M \longrightarrow \mathbb{R}^3$ an immersion with positive Gauss curvature. Then M is orientable, the Gauss map $\gamma : M \longrightarrow S^2 \subseteq \mathbb{R}^3$ is a diffeomorphism, the map $f : M \longrightarrow \mathbb{R}^3$ is an embedding and $f(M)$ is convex.

Example 3.3. For a 4-dimensional manifold M, Gauss Bonnet theorem shows that,

$$\chi(M) = (4\pi)^{-1} \int_M K d\mu.$$

If the sectional curvature $R > 0(\geq 0)$ or $< (\leq 0)$, then by a clever calculation in [C3] we will have $K > 0(\geq 0)$. Therefore, if M is a compact 4 manifold with sectional curvature $R > 0(\geq 0)$ or $< (\leq 0)$, then $\chi(M) > 0(\geq 0)$.

Of course, the next thing to do is to generalize Theorem 3.1. Naturally there are two ways to make extensions: Find a formula giving deg(γ) for all dimensions; find a formula giving $\chi(M)$ in terms of the curvature tensor for all compact Riemannian manifolds; .

In the second case, C. B.Allendoerfer and A. Well discovered the formula in 1940 [AW], and in 1944 S.S. Chern found an intrinsic proof for it. We will return to this later.

In the first case, it turns out as the following theorem [G]:

Theorem 3.4. *Let $f : N \longrightarrow \mathbb{R}^n$ be a map whose Jacobian is nonzero on the oriented boundary M of a compact $n-$manifold N. Then if x is the projection of $\mathbb{R}^n$ to some $x-$axis and $\nabla(x \circ f)$ is the gradient vector field of the composition of maps $(x \circ f)$ and Ind is its index, we have*

$$\deg(\gamma) = \chi(N) - Ind(\nabla(x \circ f)).$$

The fact that f has nonzero Jacobian on the boundary M of course means that f is an immersion on M. Since the composition $(x \circ f)$ is a map from N to the real line $\mathbb{R}$, the gradient can be defined as in advanced calculus and gives a vector field on N. The index of a vector field, which is a new term in this paper, is another topological invariant closely related to the degree of a map. In fact, for a vector field V, we can locally view it as a map from the n dimensional manifold to $\mathbb{R}^n$, and $\mathrm{Ind}(V) := \sum_{x\in\{\text{zero points of } V\}} \mathrm{Sgn}(d_x V)$. Index also has the property that if V and W are homotopic, then $\mathrm{Ind}(V) = \mathrm{Ind}(W)$. (see [Zh]-Chap11 for the details.) Then Theorem 3.2 follows easily from the Morse's beautiful equation involving the index of a vector field. Marstion Morse discovered it in 1929 in the [M].

> **Morse's Equation:** Let V be a vector field defined on N, and suppose V is not zero on the boundary M. Then $\mathrm{Ind}V + \mathrm{Ind}\partial_- V = \chi(N)$, where $\partial_- V$ is a vector field induced by V and defined on that part of boundary M where V points inside.

Brief proof of Theorem 3.2: Let $V = \nabla(x \circ f)$, then all we have to show is that $\deg(\gamma) = \mathrm{Ind}\partial_- V$. Geometrically, the direction of the gradient is the direction that the function increases most, so $W := \nabla_M(x \circ f) = \text{projection of } \nabla(x \circ f)$ onto TM. (Here, ∇_M denote the gradient in the manifold M.) And $f_*(W_p) = \pi_p(a)$, where π_p is the projection from $\mathbb{R}^{n+1}$ to $f_*(T_pM)$ and a = the unit vector along x-axis. We may suppose $\pm a$ are regular values of γ, by the Sard theorem. With the observation that,

$$\pi_p(a) = 0 \Longleftrightarrow \gamma(p) = \pm a,$$

we then ask $Z = \{q \in M : \gamma(q) = -a\}$, then $f_*(V)$ points inside at every point of Z. So it's suffice to show that $\deg(\gamma) = \mathrm{Ind}_Z f_*(W)$, since $\partial_- V$ and W is homotopic and f_* is an isomorphism. ($\mathrm{Ind}_Z f_*(W)$ means to calculate $\mathrm{Sgn}(d_x(f_*W))$ only on the subset Z of the zero points of f_*W). In fact

$$f_*(W)(x) = a - (\gamma(x) \cdot a)\gamma(x).$$

By derivation at both side, we'll have,

$$d_x(f_*(W))(\xi) = d_x\gamma(\xi), \quad \forall x \in Z.$$

By the definition of deg and Ind,

$$\deg(\gamma) = \sum_{x\in Z} \mathrm{Sgn}(d_x\gamma) = \sum_{x\in Z} \mathrm{Sgn}(d_x f_*(W)) = \mathrm{Ind}_Z f_*(W).$$

The Gauss-Bonnet theorem (Theorem 3.1) follows immediately from this theorem with a basic property of the index: If V is a vector field on an odd dimensional manifold, then $\mathrm{Ind}(-V) = -\mathrm{Ind}(V)$. If we choose the x-axis in the Theorem 3.2 to run in the opposite direction, we reverse the direction of the gradient. But, the two terms in Theorem 3.2 certainly do not care which way the x-axis is going. So we must have $\mathrm{Ind}(\nabla(x\circ f))=0$. Thus $\deg(\gamma)=\chi(N)=\frac{1}{2}\chi(M)$. The last equality follows because the Euler characteristic for an even dimensional boundary is twice the Euler characteristic of its bounded manifold. This is a fact form the long exact sequence of the homology group of the two manifolds and the Poincaré duality in Algebraic Topology.

There is one point that remains to be clarified. Does every orientable M that can be immersed in a codimension 1 Euclidean space bound an N so that the immersion can be extended to an f ? The answer is yes. This follows form a statement in [W] that the manifold is an oriented boundary if and only if all Pontrjagin numbers and all Stiefel-Whitney numbers of it are zero. In our case the Stiefel-Whitney numbers and Pontrjagin numbers of M are actually all 0 by the fact that M has a trivial normal bundle with which we take Whitney sum with TM is again trivial.

4. Higher Codimensional Case

4.1. **Generalized Theorems.** Now we are going to discuss our second way to make the extension: Find a formula giving $\chi(M)$ in terms of the curvature tensor for all manifolds. The first breakthrough is:

Theorem 4.1. *Let M^n be a closed Riemannian manifold of dimension n; let $d\mu(x)$ be the Riemannian volume-element at the point x with local coordinates x^i; let g_{ij} be the metric tensor, $g=det(g_{ij})$, $R_{i_1i_2j_1j_2}$ the Riemannian curvature tensor induced by the Levi-Civita connection at the same point; let $\varepsilon^{(i)}$ be a symbol which is equal to 1 or -1 according as $i_1,\ldots,i_n$ form an even or odd permutation of $1,\ldots,n$; and define the invariant scalar $\psi(x)$ by:*

$$\psi(x)=(2\pi)^{-n/2}(2^n(n/2)!)^{-1}\sum_{i,j}\varepsilon^{(i)}\varepsilon^{(j)}g^{-1}R_{i_1i_2j_1j_2}R_{i_3i_4j_3j_4}\ldots R_{i_{n-1}i_nj_{n-1}j_n}$$

for n even; $\psi(x)=0$, for n odd. Then,

$$\int_{M^n}\psi(x)d\mu(x)=\chi(M).$$

This is proved by C.B.Allendoerfer and A. Well in their paper [AW] in 1940, using Hermann Weyl's theory of tubes. However they required M^n in their paper to be embedded in some higher dimensional Euclidean space. Later, in the 1950's, Nash proved that every Riemannian manifold can be isometrically embedded in some Euclidean space. This finally completed the theorem above for all Riemannian manifolds. But it was not yet perfect in the sense that an intrinsic formula should have an intrinsic proof. And that was done by Chern.

Locally, we can choose a frame: $(x,e_1,\ldots,e_n)$, with $<e_i,e_j>=\delta_{ij}$. Let $(\omega_1,\ldots,\omega_n)$ be the dual basis of $(e_1,\ldots,e_n)$, and $\Omega_{i_li_{l+1}}=\Sigma_{j_lj_{l+1}}R_{i_li_{l+1}j_lj_{l+1}}\omega_{j_l}\wedge\omega_{j_{l+1}}$, $l=1,3,\ldots$

Let Ω be defined as:

$$\Omega:=(2\pi)^{-n/2}(2^n(n/2)!)^{-1}\sum\varepsilon^{(i)}\cdot\Omega_{i_1i_2}\wedge\ldots\wedge\Omega_{i_{n-1}i_n}$$

for n even, $\Omega:=0$ for n odd. Then we have,

$$\psi(x)d\mu(x)=\Omega$$

Another equivalent expression for the Gauss-Bonnet theorem is then:

Theorem 4.2. *Let M^n be a closed Riemannian manifold of dimension n, then,*

$$\int_{M^n}\Omega=\chi(M).$$

This is proved intrinsically in Chern's famous paper [C1]. Briefly speaking, his method is to construct a form Φ on the sphere bundle $\xi=\pi:E\longrightarrow M$, such that $\pi^*(\Omega)=d\Phi$. Then with Stokes's theorem, the result is easily followed.

4.2. **The Relation to the Original Gauss-Bonnet Theorem.** These two theorems look quite different from the original Gauss-Bonnet theorem, but after an easy calculation we can see that the original one is just a special case of these two when they are applied to the codimension 1 case.

Let M^n be immersed into $\mathbb{R}^{n+1}$, $\{e_i\}$ be the principal curvature vectors with principal curvatures $\{\lambda_i\}$. ([J]-3.6). By Gauss equations and the fact that the curvature tensor of $\mathbb{R}^{n+1}$ is 0, we have,

$$\begin{aligned} R_{ijlk} &= < R(e_i,e_j)e_k, e_l > \\ &= l(e_j,e_k)\cdot l(e_i,e_l) - l(e_j,e_l) \\ &= \lambda_i \cdot \lambda_j(\delta_{jk}\cdot\delta_{il} - \delta_{ik}\cdot\delta_{jl}) \end{aligned}$$

$\Longrightarrow \Omega_{ij} = 2\lambda_i\cdot\lambda_j\omega_i\wedge\omega_j$, $\Longrightarrow \Omega = v(n)^{-1}Kd\mu$. ($g$ is 1 in this case.)

So Theorem 4.1 and Theorem 4.2 imply Theorem 3.1, when M is a hypersurface. This also shows that K is intrinsic when n is even. In fact, $Kd\mu = \Omega$, which is intrinsic since Ω is independent of the choice of the frame. [C1]. By the way, when n is odd, K is not intrinsic anymore, but K^2 is intrinsic. Similarly, by choosing e_i to be the principal curvature vectors, we can show that K^2 is actually the intrinsic sum

$$\sum \varepsilon^{(i)}\varepsilon^{(j)}R_{i_1j_1i_1j_1}\ldots R_{i_nj_ni_nj_n}.$$

4.3. **The Background of the Ω and $\psi(x)$.** Ω and $\psi(x)$ in the theorems look really like a huge monster. It must exist a natural way to construct them behind the complicated expressions. In fact it turns out they are the Pfaffian of the curvature tensor of the manifold. We will make the following calculation to have it shown.

For our convenience, we define Pfaffian $Pf(A)$ for an $n\times n$ $l-$form valued matrix $A = (a_{ij})$ with $n = 2m$, by,

$$(2^m m!)Pf(A)\omega_1\wedge\ldots\wedge\omega_n = \sum \wedge_{k=1}^m (a_{i_kj_k}\wedge\omega_{i_k}\wedge\omega_{j_k}),$$

By a direct calculation,

$$Pf(A) = (2^m m!)^{-1}\sum\varepsilon^{(i)}\cdot a_{i_1i_2}\wedge\ldots\wedge a_{i_{n-1}i_n}.$$

Let $\xi = \pi : E \longrightarrow M$ be a smooth oriented n-dimensional vector bundle over M (here we even don't care what dimension M is), where $n = 2m$. If $<,>$ is a Riemannian metric for ξ, then we can consider the bundle $SO(E)$ of oriented orthonormal frames, which is a principal bundle with group $SO(n)$. We know that there is a connection ω on the principal bundle $p : SO(E) \longrightarrow M$, where ω is a matrix of 1-forms (ω_{ij}) on $SO(E)$ taking values in $o(n)$. Thus at every point of $SO(E)$ we can define the vertical part V and the horizontal part H of its tangent space. They are actually $ker(p_*)$ and $ker(\omega)$ respectively. For every k-form α on $SO(E)$, with values in a vector space V, we define a V-valued $(k+1)$-form $D\alpha$, the covariant differential of α, by

$$D\alpha(Y_1,\ldots,Y_{k+1}) = d\alpha(hY_1,\ldots,hY_{k+1})$$

where hY_i is the horizontal component of Y_i. Now we can have our curvature form $K = D\omega$, which is also a matrix of 2-forms (Ω_{ij}) with values in $o(n)$. This curvature form does indeed correspond to the curvature form we defined in the textbook [J]-3.1. In fact, there are at least five equivalent ways to define a connection. In this paper, we'll refer the other two of them besides the one defined in the textbook, i.e. the Ehresmann connection on the principal bundle and the Cartan connection on the moving frames. There is even a theory about the relations between various definitions of a connection. It turns out they all are equivalent and so are the curvature forms they induce. Interested readers may refer [S] II-Chap8 for the detail reasoning. Now we consider n-form,

$$(\pi)^{-m}Pf(K) = c\sum\varepsilon^{(i)}\cdot\Omega_{i_1i_2}\wedge\ldots\wedge\Omega_{i_{n-1}i_n}$$

where $c^{-1} = \pi^m\cdot 2^{2m}\cdot m!$. There is a unique n-form Λ on M such that $p^*(\Lambda) = (\pi)^{-m}Pf(K)$. It can be shown that Λ is closed and the cohomology class $[\Lambda]$ is independent of metric $<,>$ and of the connection ω. ([S] V-Chap13). (The reason for us to use the connection on the principal bundle is partly because it's easier to prove all kinds of properties of this Λ there.) In particular, when $\xi = TM$, this Λ is just our Ω in Theorem 4.2 by the equivalent of different definitions.

4.4. **An Outline of a Proof for Theorem 4.2 and some Insights.** We've seen that every oriented smooth bundle ξ over M of an even fiber dimension n determines a de Rham cohomology class $c(\xi) = c\cdot[\Lambda] \in H^n(M)$. When n is odd, we simply define $c(\xi) = 0$. It turns out $c(\xi)$ satisfies: ([S] V-Chap13)

(1) $c(f^*\xi) = f^*(c(\xi)) \in H^m(M_1)$, where f is smooth: $M_1 \longrightarrow M$;

(2) $c(\xi_1 \oplus \xi_2) = \frac{(m_1+m_2)!}{m_1!m_2!}c(\xi_1) \cup c(\xi_2)$, where m_i is the dimension of the bundle ξ_i;

(3) $c(\xi) = 0$, if ξ has a nowhere zero section.

But these facts practically characterize the Euler class. Indeed, a theorem in [S] V-Chap13 proved that $c(\xi)$ is a multiple of the Euler class $\chi(\xi)$. (We use the letter χ also to denote the Euler class here.) When we take ξ to be the tangent bundle, again, by a calculation in [S] V-Chap13, $c(\xi) = c\cdot\chi(\xi)$, $\forall n$.

Now, let μ to be the fundermental class of M, then we have,

$$\int_M \Omega = c^{-1}\cdot c(\xi)(\mu) = \chi(\xi),$$

which is the Gauss-Bonnet theorem!

This proof gives us several new insights into the theorem. We list three of them below:

1. The class $c(\xi)$ plays a significant role in this proof. We may try to find out what all such natural classes $c(\xi)$ are. To be precise, we define a characteristic class of dimension k for a smooth n-dimensional bundle $\xi = \pi : E \longrightarrow M$ to be an element $c(\xi) \in H^k(M)$, with the properties

(i) $c(f^*\xi) = f^*(c(\xi)) \in H^m(M_1)$, where f is smooth: $M_1 \longrightarrow M$;

(ii) $c(\xi) = c(\eta)$ if ξ is homotopic to η.

Characteristic classes are very important creatures in mathematics. When they apply to the tangent bundle and valuate on the fundamental class, they turn into the corresponding numbers. However, as a convention, we call Euler number as Euler characteristic as well. We have already applied them in our third section to prove that a manifold there can be an oriented boundary. What we've done in the fourth section is, in fact, to show that the Euler class for the tangent bundle can be expressed by the form of the curvatures.

2. Since ξ can be any vector bundle on M, it's natural to consider it as any bundle not just a tangent bundle. In particular, if M^m is a closed orientable submanifold embedded in an orientable Riemannian manifold N^{2m}, then we can consider the normal bundle $TM^\perp$. To specify the result, we need to define a new term: Gauss torsion.

With a smoothly moving frame $(P, e_1, \dots, e_n)$ attaching to every point P on N, there are forms ω_i, $\omega_{ij} = -\omega_{ji}$, Ω_{ij} according to the equations:

$$dP = \Sigma_i \omega_i e_i,$$
$$de_i = \Sigma_j \omega_{ij} e_j,$$
$$\Omega_{ij} = d\omega_{ij} - \Sigma_k \omega_{ik}\omega_{kj}.$$

It turns out that (ω_{ij}) and (Ω_{ij}) are the Cartan connection forms and curvature forms on N. We define the Gaussian torsion θ of M in N by:

$$\theta = (-1)^l \frac{1}{2^{2l}\pi^l l!}\Sigma\varepsilon^{(i)}\cdot\theta_{i_1 i_2}\wedge\dots\wedge\theta_{i_{m-1}i_m},\ \text{if } m = 2l;$$

$\theta := 0$, if m is odd. Here $\theta_{ij} = \Omega_{ij} - \Sigma_{\alpha=1}^m \omega_{i\alpha}\omega_{j\alpha}$, i, j running through $m+1$ to $2m$. Then we have the following theorem in [C2]:

Theorem 4.3. *If M^m is a closed orientable submanifold embedded in an orientable Riemannian manifold N^{2m}, then $\int_M \theta = \chi(TM^\perp)$ where θ is the Gaussian torsion of M in N.*

3. Finally, with the same thought, we can try to extend the Gauss-Bonnet theorem for manifolds with boundary. We have the following theorem in [S]:

Theorem 4.4. *Let M be a compact oriented Riemannian manifold with boundary, of even dimension n=2m, with tangent bundle $\pi : TM \longrightarrow M$, and associated sphere bundle $\pi_0 = \pi \mid S : S \longrightarrow M$. Let D be a connection on the principal bundle $p : SO(TM) \longrightarrow M$, with curvature form K, let Ω be the unique n-form on M with $p^*\Omega = (\pi)^{-m}Pf(K)$, and let Φ be an (n-1)-form on S with*

$$\pi_0^*\Omega = d\Phi,$$

Finally, let $\nu : \partial M \longrightarrow S$ be the outward pointing unit normal on ∂M. Then

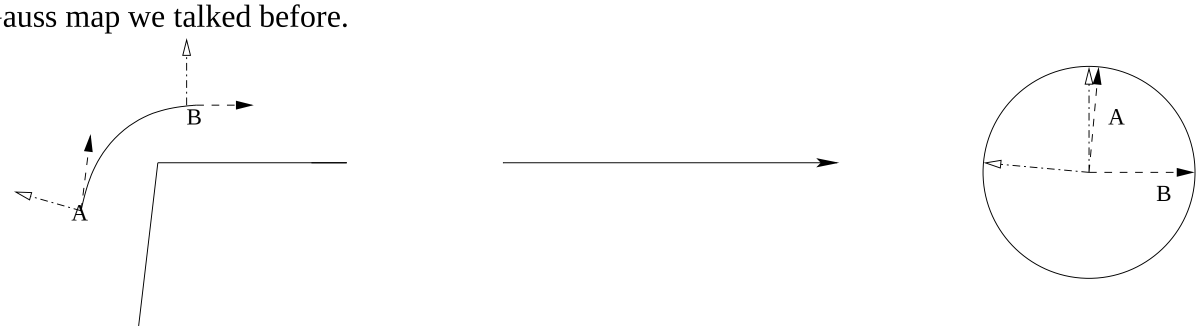
$$\int_M \Omega = \chi(M) + \int_{\partial M} \nu^*\Phi.$$

As we said before, Chern in [C1] didn't use the characteristic classes to prove the Theorem 4.2, instead he explicitly constructed a form Φ with $\pi_0^*\Omega = d\Phi$, which appears here. So it is very useful to figure this form Φ out when we are seeking the generalized Gauss-Bonnet theorem for manifolds with boundary.

5. Other Points of View

5.1. **Polyhedra.** Admittedly, all of the long story about the Gauss-Bonnet theorem above comes from its original polyhedral version. The ancient Greeks already knew its primary case: *The sum of the interior angles of a triangle equals π.* The first generalization from this primary case involves the exterior angle which is used more conveniently than the interior angle: *The sum of the exterior angles of a polygon equals 2π.*

Now, let us approximate a polygon by a smooth simple closed curve c. Consider the unit vectors tangent to c at A and B. Translate these vectors to the origin, keeping the initial and translated vectors parallel. Then the arc on S^1 cut off by the two translated vectors, i.e. the original exterior angle, represents the angle the curve has turned through. The same result follows if we replace the unit tangent vector by the unit normal vectors. And in the case of normal vectors, the map has its name as Gauss map we talked before.



Now, we have some correspondence in terminology: (denote "correspond" by ": : ")
The Gauss curvature of the curve
: : the rate of change of the tangent vector
: : the rate of change of the normal vector
: : the exterior angle,
and,
the integral of the Gauss curvature
: : the total change of the tangent vector
: : the total change of the normal vector (i.e. the degree of the Gauss map)
: : the sum of the exterior angle.

So in the polyhedral version, the Gauss curvature is naturally defined as some kind of solid angle. We'll develop the definition of the Gauss curvature and the tangential curvature (which will appear in the later theorem) in three steps:

1. Given an open convex polyhedron P in $V = \mathbb{R}^N$, we define the tangential cones of P near any point $p \in V$, and face F in P, and infinity ∞ as follows:

$$
\begin{aligned}
&C_P(p) = \{v \in V | \exists \varepsilon > 0 \text{ s.t. } p + tv \in P, \forall 0 < t < \varepsilon\},\\
&C_P(F) = C_P(q),\ \ q \in F \text{ is arbitrarily fixed},\\
&C_P(p) = \{-v \in V | \exists q \in P \text{ s.t. } q + tv \in P, \forall t > 0\}.
\end{aligned}
$$

With these cones we can associate them with their normal corn in a usual way, namely

$$C^* = \{y \in V^* | y(v) \leq 0, \forall v \in C\}.$$

2. Define the tangential curvature $\tau_P(F)$, $\tau_P(\infty)$ and Gaussian curvature $\kappa_P(F)$, $\kappa_P(\infty)$ as

$$
\begin{aligned}
&\tau_P(F) = (-1)^{\dim(F)}\mathfrak{L}_n(C_P(F) \cap B^n)/\mathfrak{L}_n(B^n),\ \ n > 0,\\
&\tau_P(\infty) = (-1)^{\dim(P)}\mathfrak{L}_n(C_P(\infty) \cap B^n)/\mathfrak{L}_n(B^n),\ \ n > 0,\\
&\tau_P(F) = (-1)^{\dim(P)-\dim(F)}\mathfrak{L}_N(C_P^*(F) \cap B^N)/\mathfrak{L}_N(B^N),\\
&\tau_P(\infty) = (-1)^{\dim(P)} - (-1)^{\dim(P)-\dim_s(C_P(\infty))}\mathfrak{L}_N(C_P^*(\infty) \cap B^N)/\mathfrak{L}_N(B^N).
\end{aligned}
$$

where B^N is the unit ball of V^* centered at the origin, the dual space V^* is identified with V. $B^n = B^N \cap C_P(P)$, and $\mathfrak{L}_N$ and $\mathfrak{L}_N$ are Lebesgue measures on V and $C_P(P)$ respectively, and $\dim_s(C_P(\infty))$ denotes the greatest dimension of subspaces of V contained in $C_P(\infty)$.

3. For every polyhedron P, we can have a decomposition $\mathfrak{D}$ of it, which is a finite collection $\{P_i\}$ of open convex polyhedra such that $P = \sqcup_i P_i$. With the decomposition, we can extend the definition of the tangential curvature and the Gaussian curvature to a polyhedron by summation:

$$
\begin{aligned}
&\tau_P(F) = \sum_{D \in \mathfrak{D} \text{ and } F \text{ is a face in } D} \tau_D(F),\\
&\tau_P(\infty) = \sum_{D \in \mathfrak{D}} \tau_D(\infty),\\
&\kappa_P(F) = \sum_{D \in \mathfrak{D} \text{ and } F \text{ is a face in } D} \kappa_D(F),\\
&\kappa_P(\infty) = \sum_{D \in \mathfrak{D}} \kappa_D(\infty).
\end{aligned}
$$

[Ch] shows that the definitions are independent of the choice of decompositions.

It turns out, in our case of polygon P, the Gauss curvature at each vertex is $\frac{1}{2\pi}$ (the exterior angle). If we denote the Gauss curvature at point p as $\kappa_{P(p)}$, then we have

$$\sum_{p \in \text{vertices of P}} \kappa_{P(p)} = 1 = \chi(\text{the closed disk}) = \chi(P).$$

This is something well-known in the ancient time. People now greatly generalize it, and make it even work for non-locally-compact P (that's where ∞ in the following formula comes out). We have,

Theorem 5.1. *For a Euclidean or Riemannian polyhedra P,*

$$\sum_{p \in \nu \cup \{\infty\}} \kappa_{P(p)} = \chi(P)$$

where ν is the set of all vertices of P. [Ch]

It is meaningful to consider such non-locally-compact polyhedra, because, in differential geometry, we have great properties for our objects—the manifolds, such as locally compactness and second countability; however, we can't expect these properties any more when we are discussing the topological ones. Moreover, the generalized Gauss-Bonnet theorem is the dual of generalized Gram-Sommerville formula:

$$\sum_{F\in\mathfrak{F}\cup\{\infty\}}\tau_P(F)=\tau_P(\infty)+\Sigma_{k=0}^{N}\tau_k(P,\mathfrak{F})=0$$

where $\mathfrak{F}$ is the collection of faces of P and $\tau_k(P,\mathfrak{F})=\Sigma_{F\in\mathfrak{F},\dim(F)=k}\tau_P(F)$. This formula also works without asking P to be locally compact. The duality is demonstrated in [Ch].

5.2. **Probability.** Since the appearance of Bismut's probabilistic proof of the local Atiyah-Singer index theorem (Bismut [B]) (for the Dirac operator on spinor bundles), there have appeared many works to reprove various forms of index theorems by probabilistic and analytic methods. And the Gauss-Bonnet theorem is one of the results.

The proof of the Gauss-Bonnet theorem is built up by three steps.

First, we construct a smooth heat kernel derived from the heat equation on the manifold, and compute its "supertrace". To be more precise, for a compact manifold M, consider the heat equation on a differential form $\alpha=\alpha(t,x)$ on M:

$$\frac{\partial\alpha}{\partial t}=\frac{1}{2}\triangle_M\alpha,\quad \alpha(0,\cdot)=\alpha_0$$

where $\triangle_M$ is the Hodge-de Rham Laplacian on $\Gamma(\Lambda^*M)$, and Λ^*M denotes the set of differential forms on M. There is a probabilistic method to solve this heat equation. ([IW], [P]) Namely, there is a smooth heat kernel

$$e^{t\triangle_M/2}(x,y):\Lambda_y^*M\longrightarrow\Lambda_x^*M$$

with respecting to the Riemannian volume measure such that the solution is given by

$$\alpha(t,x)=\int_M e^{t\triangle_M/2}(x,y)\alpha_0(y)dy$$

The heat kernel has the following two properties:

(i) $e^{t\triangle_M/2}(x,x)$: $\Lambda_x^*M\longrightarrow\Lambda_x^*M$ is a degree-preserving map,

(ii) If we define a supertrace $\Phi(T)$ for a degree-preserving linear map $T:\Lambda^*V\longrightarrow\Lambda^*V$, by

$$\Phi(T)=\sum_{p=0}^{n}(-1)^n Tr(T|_{\Lambda^pV}),$$

then,

$$\int_M\Phi\{e^{t\triangle_M/2}(x,x)\}dx=\sum_{i=0}^{\infty}e^{\mu_i t/2}\cdot\Sigma_{p=0}^{n}(-1)^p dim E_i^p,$$

where μ_i is the eigenvalue of $\triangle_M$, and E_i^p is the subspace made up by p-forms in the corresponding eigenspace of μ_i.

Second, we apply Hodge theorem to express the Euler characteristic $\chi(M)$ by the supertrace of the heat kernel. In fact, by Hodge theory, the eigenvalues of $\triangle_M=\mu_0=0>\mu_1>\mu_2\ldots$, and each eigenspace E_i is finite dimensional with the direct decomposition,

$$E_i=E_i^0\oplus\ldots\oplus E_i^n$$

where E_i^P is made up by p-forms. Again by Hodge theory [MS],

$$\begin{aligned}\Sigma_{p=0}^{n}(-1)^p\dim E_i^p\quad &=0,\, if\,\mu_i<0\\ &=\Sigma_{i=0}^{n}(-1)^p\dim(H^p(M))=\chi(M),\, if\,\mu_i=0.\end{aligned}$$

Hence,

$$\chi(M) = \int_M \Phi\{e^{t\triangle_M/2}(x,x)\}dx$$

Finally, we claim that this heat-kernel integrand can be identified with the curvature integrand in the Gauss-Bonnet theorem. This is what I will not demonstrate here, interested readers may refer to [H]. Moreover [INS] and [Gi] went forward and worked it out for the manifolds with boundary.

Combining these three steps, the Gauss-Bonnet theorem is proven.

6. Relations with Physics

From the viewpoint of physics, the optical Berry phase is a direct result of the Gauss-Bonnet theorem.

Berry phase was first pointed out by Berry in 1984 [Be]. A quantum system described by a Hamiltonian which depends on cyclic time-dependent adiabatic parameters acquires a phase over and above the dynamical phase. This extra phase is generally known as the Berry phase.

It was pointed out by Chiao and Wu [CW], that there is an optical version for it. Linearly polarized laser light is fed into a single optical fiber which is wound N times round a cylinder, making a helix. If the initial and final directions of the fiber are identical, the photon momentum vector k describes a closed path. If k is of constant magnitude this path is a closed circuit on the surface of a sphere. This circuit C will subtend a solid angle $\Omega(C)$ at the center of the sphere and Chiao and Wu surmised that, the change in phase of the laser light would be given by

$$\gamma(C) = -\sigma\Omega(C)$$

where $\sigma(=\pm 1)$ is the photon helicity and

$$\Omega(C) = 2\pi N(1 - cos\theta)$$

where θ is the pitch angle of the helix.

They further pointed out that in this experiment the phase change $\gamma(C)$ implies that the plane of polarization of the light is rotated through an angle $\gamma(C)$; so the Berry phase is, in this case, and angle of optical rotation. This predicted rotation was subsequently observed by Tomita and Chiao [TC].

Moreover, L. H. Ryder shows in [R] that the change in phase $\gamma(C) = -\sigma(2\pi - \oint_C K_1 d\mu_C)$, where K_1 denotes the Gauss curvature on C. With this we can prove the optical Berry phase geometrically by the Gauss-Bonnet theorem.

Let C be a closed smooth path (in particular it can be the circuit), D be the surface in S^2 enclosed by C, then by the Gauss-Bonnet theorem for manifolds with boundary,

$$2\pi = \int\int_D K d\mu_D + \oint_C K_1 d\mu_C$$

where K and K_1 denote the Gauss curvature on D and C respectively.

It's not hard to find that $\int\int_D K d\mu_D = \text{area}(D) = \Omega(C)$, hence, with the equation L.H.Ryder shows, we have,

$$\gamma(C) = -\sigma\Omega(C).$$

And this is what was surmised by Chiao and Wu and found by Tomita and Chiao.

7. References

[AW] C. B. Allendoerfer and A. Well, *The Gauss-Bonnet Theorem for Riemannian Polyhedra*, Trans. Amer. Math. Soc. 53 (1943), 101-129.

[B] J. M. Bismut, *The Atiyah-Singer theorems: a Probabilistic Approach*, J. of Func. Anal. 57 (1984), I, 56-99; II, 329-348.

[Be] M. V. Berry, Proc. R. Soc. A 392 (1984), 45.

[Ch] B. Chen, *The Gram-Sommerville and Gauss-Bonnet Theorem and Combinatorial Geometric Measures for Non-compact Polyhedra*, Adv. in Math. 91 (1992), 269-291.

[C1] S. S. Chern, *A Simple Intrinsic Proof of the Gauss-Bonnet Formula for Closed Riemannian Manifolds*, Annals, 45 (1944), 747-752.

[C2] S. S. Chern, *On the Curvature Integra in a Riemannian Manifold*, Ann. of Math., 46 (1945), 674-684.

[C3] S. S. Chern, *On Curvature and Characteristic Classes of a Riemannian Manifolds*, Abh. Math. Sem. Univ. Hamburg 20 (1955), 117-126.

[CW] R. Y. Chiao and Y. S. Wu,Phys. Rev. Lett. 57 (1986), 993.

[G] D. H. Gottlieb, *All the Way with Gauss-Bonnet and the Sociology of Mathematics*, American Mathematical Monthly, 104 (N6) (Jun-Jul, 1996), 467-469.

[Gi] P. B. Gilkey, *The Boundary Intergrand in the Formula for the Signature and Euler Characteristic of a Riemannian Manifold with Boundary*, Anv. in Math., 15 (1975), 334-360.

[H] E. P. Hsu, *Stochastic Local Gauss-Bonnet-Chern Theorem* Journal of Theoretical Probability, 10 (1997), 819.

[J] J. Jost, *Riemannian Geometry and Geometric Analysis*, Springer-Verlag, 1995.

[M] M. Morse, *Singular Points of Vector Fields under General Boundary Conditions,* Amer. J. Math. 51 (1929), 165-178.

[MS] H. P. Mackean and I. M. Singer, *Curvature and the Eigenvalues of the Laplacian*, J. Differential Geometry 1 (1967), 43-69.

[IW] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes,* Kodansha/North-Holland, Tokyo/Amsterdam, 1981, Second Ed., 1989.

[P] P. Malliavin, *Géométrie Différentielle Stochastique*, Les Presses de l'Université de Montréal, 1978.

[R] L. H. Ryder, *The optical Berry Phase and the Gauss-Bonnet Theorem*, Eur. J. Phys. 12 (1991), 15-18.

[S] Spivak, *A Comprehensive Introduction to Differential Geometry*.

[SUW] I. Shigekawa, N. Ueki and S. Watanabe, *A Probabilistic Proof of the Gauss-Bonnet-Chern Theorem for Manifolds with Boundary Osaka J. Math.* 26 (1989), 897-930.

[TC] A. Tomita and R. Y. Chiao, *Phys. Rev. Lett.* 57 (1986), 937.

[W] C. T. C. Wall, *Determination of the Cobordism Ring*, Annals of Math. 72 (1960), 262-311.

[Zh] Z. S. Zhang, *Lectures in Differential Topology, in Chinese*, 1996.