Every set has a least jump enumeration

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Abstract

Given a computably enumerable set B, there is a Turing degree which is the least jump of any set in which B is computably enumerable, namely $\boldsymbol{0}'$. Remarkably, this is not a phenomenon of computably enumerable sets. We show that for *every* subset A of $\mathbb{N}$, there is a Turing degree, $\boldsymbol{c}'_\mu(A)$, which is the least degree of the jumps of all sets X for which A is $\Sigma^0_1(X)$.

1 Introduction

In computability theory there are two fundamental notions, relative computability and relative computable enumerability. The discovery of the existence of sets that are computably enumerable but not computable initiated a detailed study of the Turing universe, particularly with respect to relative computability.

The fundamental operator in computability theory is the jump operator, so named because it raises the information content of a set, that is $A <_T A'$. From the point of view of arithmetical complexity this operator can be thought of as adding an existential quantifier to the complexity of A, namely A' is $\Sigma^0_1(A)$.

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In this paper we are concerned with the relationship between relative computability and relative computable enumerability of subsets of $\mathbb{N}$. In particular, given a set A, A can be enumerated from many sets X. For instance if A is a computably enumerable set, then it can be enumerated from the empty set and hence from all sets X. Therefore it is natural to seek to understand the set

$$\mathcal{C}(A) = \{X \mid A \in \Sigma^0_1(X)\},$$

for sets A, namely the set of sets from which A can be enumerated. We can also consider the degrees of members of this set. For example if A is the complement of a computably enumerable set then $\deg(A)$ is the least Turing degree of members of $\mathcal{C}(A)$. However, there are sets A for which $\mathcal{C}(A)$ has no member of least Turing degree. In [3], using a finite extension argument, Richter proves that

Theorem 1 *There is a non-c.e. set A such that A is c.e. in two sets B and C with B and C forming a minimal pair.*

This yields the desired result.

Corollary 2 *There is a set A such that $\mathcal{C}(A) = \{X \mid A \in \Sigma^0_1(X)\}$ has no member of least Turing degree.*

Proof: Take A, B and C as in Theorem 1. Then A cannot be Σ^0_1 in a set D for which $D \leqslant_T B$ and $D \leqslant_T C$, because B and C form a minimal pair, implying that D is computable. This contradicts A being a non-c.e. set. ∎

So not every set is computably enumerable in some set of least Turing-degree. In fact it is possible to strengthen Richter's result to make A of non-c.e. degree and we sketch a proof of this in section 2.

With our main result we show that all pathologies disappear when one looks at the 1-jump version of $\mathcal{C}(A)$. We define

$$\mathcal{C}'(A) = \{X' \mid A \in \Sigma^0_1(X)\}.$$

We prove the following result.

The Main Theorem *For all $A \subseteq \mathbb{N}$ there is an $F \in \mathcal{C}(A)$ such that for all $X \in \mathcal{C}(A)$, $F' \leqslant_T X'$.*

That is for all sets A, $\mathcal{C}'(A)$ always has a member of least Turing degree. We denote this least degree by $\boldsymbol{c}'_\mu(A)$. We can immediately make the following observations from the existence of $\boldsymbol{c}'_\mu(A)$ for all sets A.

1. $\boldsymbol{c}'_\mu(A) \leqslant \deg_T(A')$.

2. For any computably enumerable set W, $\boldsymbol{c}'_\mu(W) = \boldsymbol{0}'$.

3. For any low set L, $\boldsymbol{c}'_\mu(L) = \boldsymbol{0}'$.

4. If $A = B \oplus \overline{B}$ then $A \in \Sigma^0_1(X)$ iff $A \leqslant_T X$, and so it follows that $A' \in \boldsymbol{c}'_\mu(A)$.

5. If A is a 1-generic set (or in fact any $\boldsymbol{GL_1}$ set) then $A \in \Sigma^0_1(X)$ implies that $A' \leqslant_T X'$, and therefore it follows from (1) that $A' \in \boldsymbol{c}'_\mu(A)$.

6. Soare and Stob [5] have shown that above every non-zero c.e. degree $\boldsymbol{a}$ there is a non-c.e. degree which is c.e. in $\boldsymbol{a}$. It follows that above every low computably enumerable set L there is a non-computably enumerable set A such that $\boldsymbol{c}'_\mu(A) = \boldsymbol{0}'$.

7. We have an inversion theorem as follows. Let $\boldsymbol{a} \geqslant \boldsymbol{0}'$. Then by the Friedberg Jump Inversion theorem there is a degree $\boldsymbol{b}$ such that $\boldsymbol{b}' = \boldsymbol{a}$. Suppose $B \in \boldsymbol{b}$ and consider the set $B \oplus \overline{B} \in \boldsymbol{b}$. Then $\boldsymbol{c}'_\mu(B \oplus \overline{B}) = \boldsymbol{a}$.

Towards generalising the main theorem we make the following definition.

Definition 3 Suppose A is a subset of $\mathbb{N}$. Define

$$\mathcal{C}^{(n)}(A) = \{X^{(n)} \mid A \in \Sigma^0_n(X)\}.$$

We denote the least Turing degree of members of $\mathcal{C}^{(n)}(A)$ by $\boldsymbol{c}^{(n)}_\mu(A)$, if it exists.

Consider

$$\mathcal{C}^{(n+1)}(A) = \{X^{(n+1)} \mid A \in \Sigma^0_{n+1}(X)\}.$$

Now if A is $\Sigma^0_{n+1}(X)$ for some set X, then A must be $\Sigma^0_n(X')$. Hence

$$\begin{aligned}\mathcal{C}^{(n+1)}(A) &= \{X^{(n+1)} \mid A \text{ is } \Sigma^0_n(X')\}\\ &= \{(X')^{(n)} \mid A \text{ is } \Sigma^0_n(X')\}\\ &\subseteq \mathcal{C}^{(n)}(A).\end{aligned}$$

The main theorem extends as follows.

Theorem 4 *For all $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, $\boldsymbol{c}_\mu^{(n)}(A)$ exists.*

So $\mathcal{C}^{(n)}(A)$ always has a member of least Turing degree and we prove this in section 3.

When studying effective versions of classical mathematics, a natural way to assign a degree of unsolvability to a structure, such as a linear order or a group, is to take the least degree of presentations B isomorphic to the given structure A. However a least degree need not always exist. The motivation for studying $\mathcal{C}'(A)$ arose from an observation of Downey and Jockusch in [1] related to presentations of torsion free abelian groups. In section 4 we make an application of the main theorem to a question of Downey and Jockusch.

2 Extending A Theorem Of Richter

We briefly sketch a proof of the strengthened theorem of Richter mentioned in the introduction.

Theorem 5 *There is a d-c.e. set A of non-c.e. degree and sets B and C such that A is $\Sigma^0_1(B)$ and $\Sigma^0_1(C)$, and B and C form a minimal pair.*

We must meet the following requirements for all $e, i, j \in \mathbb{N}$ where V is a computably enumerable operator.

$$\mathrm{P} : A = V(B) \,\&\, A = V(C),$$

$$\mathrm{N}_e : \Phi_e(B) = \Phi_e(C) = g \text{ total} \implies g \text{ is computable},$$

$$\mathrm{R}_{e,i,j} : A \neq \Phi_i(W_e) \vee W_e \neq \Psi_j(A).$$

All the strategies are the obvious ones. To meet the P requirement we enumerate V-axioms of the form $\langle x, b(x,s)\rangle$ for $x \in A$ and ensure $b(x,s) \in B$, and similarly for C. To meet an $\mathrm{R}_{e,i,j}$-requirement we use the Cooper-Lachlan construction of a set of properly d-c.e. degree. The only interesting point to discuss is the following conflict between a minimal pair requirement N of higher priority than an $\mathrm{R}_{e,i,j}$-requirement.

Activity for an $\mathrm{R}_{e,i,j}$-requirement requires the extraction of $b(x, s+1)$ from B and $c(x, s+1)$ from C. This conflicts with the N-requirement wanting to preserve at least one of B and C. Suppose then that N is of higher priority than $\mathrm{R}_{e,i,j}$. Also suppose we extract x from A at stage $s+1$ for the sake of $\mathrm{R}_{e,i,j}$ and hence extract $b(x,s)$ from B and $c(x,s)$ from C. If we see a computation

$$\Phi(B)(y)\!\downarrow\![t+1] \neq \Phi(B)(y)\!\downarrow\![s+1]$$

for some expansionary stage $t+1 > s+1$, then we can return B to its configuration at stage $s+1$ on the use of the computation $\Phi(B)(y)\!\downarrow\![s+1]$ when

$$\Phi(B)(y)\!\downarrow\![s+1] = \Phi(C)(y)\!\downarrow\![s+1] \neq \Phi(C)(y)\!\downarrow\![t+1] = \Phi(B)(y)\!\downarrow\![t+1].$$

This will involve enumerating/extracting various $b(z,s)$ into/from B at stage $t+1$. Such action could injure the P requirement as there may now be some z with $A(z)[t+1] \neq V(B)(z)[t+1]$. Therefore we correct the P requirement by enumerating into A those $z \in V(B)[t+1]$ that are currently not in A, and also correct $V(C)$ by enumerating new axioms $\langle z, c(z,t+1)\rangle \in V[t+1]$ with a suitably large new $c(z,t+1)$. For those $z \notin V(B)[t+1]$ that are currently in A we may not be able to correct via A-extraction of z because this may then require C-extractions injuring the computation $\Phi(C)(y)\!\downarrow\![t+1]$. Therefore for those $z \in A[t] - V(B)[t+1]$ we enumerate new B-axioms with suitably large new $b(z,t+1)$. This action wins the N requirement via a diagonalisation, maintains a correct V for the P-requirement, but potentially loses $\mathrm{R}_{e,i,j}$. However this strategy will only present finite injury to $\mathrm{R}_{e,i,j}$.

It should now be clear how to carry out a construction to meet all the requirements and hence prove the theorem.

3 The Proof Of The Main Theorem

We now turn to the proof of the Main Theorem and its generalisation. Suppose A is a subset of $\mathbb{N}$. Recall we define

$$\mathcal{C}^{(n)}(A) = \{X^{(n)} \mid A \in \Sigma^0_n(X)\},$$

and denote the least Turing degree of the members of $\mathcal{C}^{(n)}(A)$ by $\boldsymbol{c}^{(n)}_\mu(A)$, if it exists.

In the sequel we will prove Theorem 4, namely that for all $A \subseteq \mathbb{N}$, $\boldsymbol{c}^{(n)}_\mu(A)$ exists for all $n \in \mathbb{N}$. However we begin by proving the special case $n = 1$ to motivate the proof of the full theorem.

The Main Theorem *For all $A \subseteq \mathbb{N}$ there is an $F \in \mathcal{C}(A)$ such that for all $X \in \mathcal{C}(A)$, $F' \leqslant_T X'$.*

Proof: Given a set A we want to construct a set F such that A is $\Sigma_1^0(F)$ and $F' \leqslant_T X'$ for all sets $X \in \mathcal{C}(A)$. Originally the first two authors proved the theorem using a full approximation construction. The third author subsequently discovered an elegant proof using forcing with finite conditions which is the proof we present below.

For background on forcing arguments in computability theory we direct the reader to Lerman [2]. We now define our notion of forcing.

Definition 6 Let $X \subseteq \mathbb{N}$.

- The partial order $\boldsymbol{P}_1^X$ is the set of finite enumerations of subsets of X ordered by extension.

  So an element of $\boldsymbol{P}_1^X$ is a function, p say, from $\{0, 1, \ldots, k\}$ into X, for some $k \in \omega$.

- For p and q in $\boldsymbol{P}_1^X$, we say p extends q in $\boldsymbol{P}_1^X$ and write $p >_1^X q$, if the graph of p is contained in the graph of q.

- We use $\mathbf{1}$ to denote the empty function, and of course $\mathbf{1} \in \boldsymbol{P}_1^X$.

We fix a $\boldsymbol{P}_1^A$ generic enumeration G_1 of A. Clearly A is $\Sigma_1^0(G_1)$ since $y \in A$ if and only if there is an x such that $(x, y) \in G_1$.

Let F be the forcing relation for $\Pi_1^0(\mathcal{G}_1)$ sentences. That is,

$$F = \{(p, \varphi) \mid p \notin \boldsymbol{P}_1^A \text{ or } (\varphi \in \Pi_1^0(\mathcal{G}_1) \text{ and } p \nVdash_1^A \varphi)\}.$$

We first observe that $A \leqslant_T F$ as follows. Let e be an index such that $\Phi_e(Y)(x)\uparrow$ for all oracles Y and all $x \in \mathbb{N}$. Let φ be the $\Pi_1^0(\mathcal{G}_1)$ sentence $\forall x(\Phi_e(\mathcal{G}_1)(x)\uparrow)$. Then $(p, \varphi) \in F$ if and only if p is not a forcing condition. Now, let q_n be the constant function from the set $\{0\}$ to n. Then q_n is a forcing condition if and only if $n \in A$. Hence $n \in A$ if and only if $(q_n, \varphi) \notin F$.

We now prove two lemmas to show that F witnesses the existence of $\boldsymbol{c}'_\mu(A)$.

Lemma 7 *$F \leqslant_T X'$ for all $X \in \mathcal{C}(A)$.*

Proof: For any p and φ, $(p, \varphi) \in F$ if and only if either

1. $p \notin \boldsymbol{P}_1^A$, or

2. there is a proper extension q of p in $\boldsymbol{P}_1^A$ and an $x \in \mathbb{N}$ such that $q \Vdash_1^A \neg\varphi(x)$.

Since $\boldsymbol{P}_1^A \leqslant_T A$, in the first case we have that $p \in \boldsymbol{P}_1^A$ is computable in X' for any set X for which A is $\Sigma_1^0(X)$. In the second case, note that $\neg\varphi$ is a bounded formula, and that we have an existential condition that refers to a finite amount of positive information about A. This condition is $\Sigma_1^0(X)$ for sets X for which A is $\Sigma_1^0(X)$, and hence for such an X, case 2 is computable in X'. Therefore $F \leqslant_T X'$ for any X such that A is $\Sigma_1^0(X)$. ∎

Lemma 8 *There is a set G such that A is $\Sigma_1^0(G)$ and $G' \leqslant_T F$.*

Proof: We construct $G \leqslant_T F$ so that every fact about G' is forced in the sense of $\boldsymbol{P}_1^A$.

Step *0*: Let $p_0 = \mathbf{1}$.

Step $\boldsymbol{s+1=2e}$:

1. If $(p_s, \varphi) \in F$, then let p_{s+1} be the least extension of p_s in $\boldsymbol{P}_1^A$ such that $p_{s+1} \Vdash_1^A \neg\varphi$.

2. Otherwise, $(p_s, \varphi) \notin F$. Let $p_{s+1} = p_s$ since $p_s \Vdash_1^A \varphi$.

Step $\boldsymbol{s+1=2e+1}$:
Let x be the least number in A not in the range of p_s. Let p_{s+1} be the least extension of p_s with x in its range.
At the end of the construction define $G = \bigcup_{s\in\mathbb{N}} p_s$.

Now clearly A is $\Sigma_1^0(G)$ since $y \in A$ if and only if there is an x such that $(x, y) \in G$. Observe that for case 1 of the even stages of the construction, we can find p_{s+1} computably in A. Hence $A \oplus F$ is an oracle that can decide every Π_1^0 condition about G. Therefore $G' \leqslant_T A \oplus F$.
We have already seen that $A \leqslant_T F$ and hence $G' \leqslant_T F$. Consequently there is a set G such that A is $\Sigma_1^0(G)$ and $G' \leqslant_T F$. ∎

By Lemmas 7 and 8 $\boldsymbol{c}'_\mu(A)$ exists and is $\deg_T(F)$. This completes the proof of the Main Theorem. ∎

We now prove the general result as stated in Theorem 4.

Theorem 4 *For all $n \in \mathbb{N}$, $\mathcal{C}^{(n)}(A) = \{X^{(n)} \mid A \text{ is } \Sigma_n^0(X)\}$ has a member of least Turing degree.*

Proof: Towards a generalisation of our notion of forcing in the $n = 1$ case we make the following definition.

Definition 9 For a set $X \subseteq \mathbb{N}$, we define the partial order $\boldsymbol{P}_m^X$ by induction on m.

- $\boldsymbol{P}_1^X$ is defined as in Definition 6, namely as the set of finite enumerations of subsets of X, ordered by extension. So a condition in $\boldsymbol{P}_1^X$ is given by a function p such that the domain of p is of the form $\{0, 1, \dots, k\}$ and its range is a subset of X; for p and q in $\boldsymbol{P}_1^X$, $p >_1^X q$ if the graph of p is contained in the graph of q.

- An element of $\boldsymbol{P}_m^X$ is a sequence $(p_m, p_{m-1}, \dots, p_1)$ such that

  - $p_m \in \boldsymbol{P}_1^X$ and
  - for all i less than m, $p_i \in \boldsymbol{P}_1^{\overline{p_{i+1}}}$, where we regard the graph of p_{i+1} as a subset of $\mathbb{N}$ and $\overline{p_{i+1}}$ as the complement of the graph of p_{i+1}.

  For $\overrightarrow{p}$ and $\overrightarrow{q}$ in $\boldsymbol{P}_m^X$, $\overrightarrow{p} >_m^X \overrightarrow{q}$ if for each i less than or equal to m, the graph of p_i is contained in the graph of q_i.

So for a set $X \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, our notion of forcing is $\langle \boldsymbol{P}_m^X, \leqslant_m^X \rangle$. A $\boldsymbol{P}_m^X$-generic filter is Turing equivalent to a sequence $(G_m, G_{m-1}, \dots, G_1)$ such that G_m is a function from $\mathbb{N}$ onto X and for all i less than m, G_i is a function from $\mathbb{N}$ onto the complement of the graph of G_{i+1}. We will identify the generic filter with its associated sequence of functions. We also introduce the following notation to aid the presentation of the forcing conditions (sequences) in $\boldsymbol{P}_m^X$.

Notation 10

- We use $\mathbf{1}$ to indicate the empty function.

- We use $\overrightarrow{p_m}^X$ to indicate an element of $\boldsymbol{P}_m^X$, but we will often simply write $\overrightarrow{p_m}$ if it is clear to which set of conditions it belongs. We write $(p_m, \dots, p_{k+1}, \overrightarrow{q_k})$, to denote that $\overrightarrow{q_k}$ is an element of $\boldsymbol{P}_k^{\overline{p_{k+1}}}$. Similarly, if we write $(\overrightarrow{\mathbf{1}}, \overrightarrow{q_k})$ for an element of $\boldsymbol{P}_m^X$, then we are referring to the element $\overrightarrow{p_m}$ such that for all i between m and $k+1$, $p_i = \mathbf{1}$, and for all i between k and 1, $p_i = q_i$. Given $\overrightarrow{p_m}^X$, we use $\overrightarrow{p_k}^Y$ to indicate the sequence consisting of the last k elements of $\overrightarrow{p_m}^X$ viewed as an element of $\boldsymbol{P}_k^Y$. Of course, this notation can only be applied when $\overrightarrow{p_k} \in \boldsymbol{P}_k^Y$.

- We use $\mathcal{G}^X$ as the symbol in the forcing language which refers to the generic filter with respect to a set X. We use $\mathcal{G}^X_k$ as the symbol in the forcing language which refers to the kth function associated with the generic filter $\mathcal{G}^X$.

Before specializing to the set A of Theorem 4, we prove two lemmas whose purpose is to establish the arithmetical complexity of the relation $\overrightarrow{p_m} \Vdash^X_m \varphi$. for $\Pi^0_n(\mathcal{G}^X_1)$ sentences φ.

Lemma 11 *Suppose φ is a $\Pi^0_n(\mathcal{G}^X_1)$ sentence, $m \geqslant n+1$, and $\overrightarrow{p_m} \Vdash^X_m \varphi$. Then for every Z, if $\overrightarrow{p_m}^Z \in \boldsymbol{P}^Z_m$ then $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_{n+1}}) \Vdash^Z_m \varphi$.*

Proof: Let Z be a subset of $\mathbb{N}$. We proceed by induction on n. We treat the cases when n is either 1 or 2 directly.

First, $n = 1$. Suppose that φ is $(\forall x)\varphi_0(x, \mathcal{G}^X_1)$, φ_0 is a $\Pi^0_0(\mathcal{G}^X_1)$ formula, that $\overrightarrow{p_m} \Vdash^X_m \varphi$ and $\overrightarrow{p_m}^Z \in \boldsymbol{P}^Z_m$. Then for every $q_1 \in \boldsymbol{P}^{\overline{p_2}}_1$ and for every $x \in \mathbb{N}$ it cannot be the case that $\neg\varphi_0(x, q_1)$. (Otherwise $(p_m, \ldots, p_2, q_1)$ would extend $\overrightarrow{p_m}$ and force the negation of φ.) Consequently, there cannot be an $x \in \mathbb{N}$, a Z contained in $\mathbb{N}$, and an extension $\overrightarrow{q}$ of $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_2})$ in $\boldsymbol{P}^Z_m$ such that $\overrightarrow{q} \Vdash^Z_m \neg\varphi_0(x, \mathcal{G}^X_1)$. So, $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_2}) \Vdash^Z_m \varphi$, as required.

Second, $n = 2$. As above, suppose that φ is $(\forall x)(\exists y)\varphi_0(x, y, \mathcal{G}^X_1)$, that $\overrightarrow{p_m} \Vdash^X_m \varphi$ and that $\overrightarrow{p_m}^Z \in \boldsymbol{P}^Z_m$. To show $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_3}) \Vdash^Z_m \varphi$, let $\overrightarrow{r_m}^Z$ be a condition extending $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_3})$ in $\boldsymbol{P}^Z_m$. Then $(p_m, p_{m-1}, \ldots, p_3, \overrightarrow{r_2})$ extends $\overrightarrow{p_m}$ in $\boldsymbol{P}^X_m$. (Note that $(p_m, p_{m-1}, \ldots, p_4, \overrightarrow{r_3})$ might not be a condition.) Let x be an element of $\mathbb{N}$. Since $\overrightarrow{p_m} \Vdash^X_m \varphi$, there is a y_0 and there is a condition $\overrightarrow{q_m}$ extending $(p_m, p_{m-1}, \ldots, p_3, \overrightarrow{r_2})$ in $\boldsymbol{P}^X_m$ such that $\overrightarrow{q_m} \Vdash^X_m \varphi_0(x, y_0, \mathcal{G}^X_1)$. Further, we may assume that the bounds on the quantifiers in $\varphi_0(x, y_0, \mathcal{G}^X_1)$ are less than the domain of q_1. Consequently, for every function G_1 extending q_1, $\varphi_0(x, y_0, G_1)$ holds. In turn this implies $(r_m, \ldots, r_2, q_1) \Vdash^Z_m \varphi_0(x, y_0, \mathcal{G}^X_1)$, and so $\overrightarrow{r_m}$ could not force $(\forall y)\neg\varphi_0(x, y, \mathcal{G}^X_1)$. Since no extension of $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_3})$ can force a counterexample to the truth of $(\forall x)(\exists y)\varphi_0(x, y, \mathcal{G}^X_1)$, we are able to conclude $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_3}) \Vdash^Z_m \varphi$.

Now for the inductive case, which is modeled on the case of $n = 2$ using induction in place of the fact that forcing is the same as truth relative to all generic filters. Suppose φ is a $\Pi^0_n(\mathcal{G}^X_1)$ sentence $(\forall x)(\exists y)\varphi_{n-2}$ with φ_{n-2} a $\Pi^0_{n-2}(\mathcal{G}^X_1)$ formula, and suppose that $\overrightarrow{p_m} \Vdash^X_m \varphi$ and $\overrightarrow{p_m}^Z \in \boldsymbol{P}^Z_m$.

To show that $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_{n+1}}) \Vdash^Z_m \varphi$, let $\overrightarrow{r_m}$ be a condition extending $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_{n+1}})$ in $\boldsymbol{P}^Z_m$. Note that $(p_m, p_{m-1}, \dots, p_{n+1}, \overrightarrow{r_n})$ extends $\overrightarrow{p_m}$ in $\boldsymbol{P}^X_m$. Let x be in $\mathbb{N}$. Since $\overrightarrow{p_m} \Vdash^X_m \varphi$, there is a y_0 and there is a condition $\overrightarrow{q_m}$ extending $(p_m, p_{m-1}, \dots, p_{n+1}, \overrightarrow{r_n})$ in $\boldsymbol{P}^X_m$ such that $\overrightarrow{q_m} \Vdash^X_m \varphi_{n-2}(x, y_0)$. By induction, $(\overrightarrow{\mathbf{1}}, \overrightarrow{q_{n-1}}) \Vdash^Z_m \varphi_{n-2}(x, y_0)$. Now note that $(r_m, r_{m-1}, \dots, r_n, \overrightarrow{q_{n-1}})$ extends $\overrightarrow{r_m}$ in $\boldsymbol{P}^Z_m$. Consequently, $\overrightarrow{r_m}$ cannot force x to be a witness for the negation of φ. Since no extension of $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_{n+1}})$ can force the negation of φ in $\boldsymbol{P}^Z_m$, we may conclude that $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_{n+1}}) \Vdash^Z_m \varphi$. ■

Lemma 12 *For φ a $\Pi^0_n(\mathcal{G}^Z_1)$ sentence, $m \geqslant n+1$, and $\overrightarrow{p_m}^Z \in \boldsymbol{P}^Z_m$, the relation $\overrightarrow{p_m} \Vdash^Z_m \varphi$ is Π^0_n.*

Proof: Let φ be a $\Pi^0_n(\mathcal{G}^Z_1)$ sentence, $m \geqslant n+1$, and $\overrightarrow{p_m}^Z \in \boldsymbol{P}^Z_m$.

Then by Lemma 11 relative to Z, $\overrightarrow{p_m} \Vdash^Z_m \varphi$ if and only if $(\overrightarrow{\mathbf{1}}, \overrightarrow{p_{n+1}}) \Vdash^\emptyset_m \varphi$. (Here $\overrightarrow{\mathbf{1}}$ may be empty.) Note that $\boldsymbol{P}^\emptyset_m$ is a computable partial order. The usual analysis of forcing shows that a condition's forcing a Π^0_n sentence in $\boldsymbol{P}^\emptyset_m$ is a Π^0_n condition. ■

We are now in a position to specialize to A and prove lemmas for a general $n \in \mathbb{N}$ in an analogous fashion to the lemmas for $n = 1$. In the $n = 1$ case we saw that $A \in \Sigma^0_1(G_1)$ where G_1 was $\boldsymbol{P}^A_1$ generic. For a given $n \in \mathbb{N}$, the analogous fact is as follows.

Lemma 13 *If $G = (G_n, \dots, G_1)$ is $\boldsymbol{P}^A_n$ generic, then A is Σ^0_n relative to G_1.*

Proof: Proceed by induction on n.

When n is equal to 1, G is (G_1), and G_1 maps $\mathbb{N}$ onto A. Consequently, A is computably enumerable in G_1 and hence $\Sigma^0_1(G_1)$.

For n greater than 1, say G is $(G_n, G_{n-1}, \dots, G_1)$ as indicated earlier. By an inductive argument, the complement of G_n is Σ^0_{n-1} in G_1, and so G_n is $\Pi^0_{n-1}(G_1)$. But then $y \in A$ if and only if there is an x such that $(x, y) \in G_n$, and so A is $\Sigma^0_n(G_1)$. ■

Now define

$$F_n = \{(\overrightarrow{p_n}^A, \varphi) : \overrightarrow{p_n} \notin \boldsymbol{P}^A_n \text{ or } (\varphi \in \Pi^0_n(\mathcal{G}^A_1) \text{ and } \overrightarrow{p_n} \nVdash^A_n \varphi)\}.$$

We now prove the analogous lemma to Lemma 7.

Lemma 14 *If A is $\Sigma^0_n(X)$ then F_n is computable in $X^{(n)}$.*

Proof: Since A is $\Sigma^0_n(X)$ and $\overrightarrow{p_n}$'s belonging to $\boldsymbol{P}^A_n$ depends only on finitely much positive information about A, $\boldsymbol{P}^A_n$ is $\Sigma^0_n(X)$. Thus, whether a sequence $\overrightarrow{p_n}$ is an element of $\boldsymbol{P}^A_n$ is computable in $X^{(n)}$.

Given that $\overrightarrow{p_n} \in \boldsymbol{P}^A_n$ and $\varphi = (\forall x)\varphi_{n-1}$ is $\Pi^0_n(\mathcal{G}^A_1)$, $\overrightarrow{p_n}$ fails to force φ if and only if there is an x and an extension $\overrightarrow{q_n}$ of $\overrightarrow{p_n}$ in $\boldsymbol{P}^A_n$ such that $\overrightarrow{q_n} \Vdash^A_n \neg\varphi_{n-1}(x)$. By Lemma 12, whether $\overrightarrow{q_n} \Vdash^A_n \neg\varphi_{n-1}(x)$ is a Π^0_{n-1} condition on q_n. Consequently, $\overrightarrow{p_n}$ fails to force φ in $\boldsymbol{P}^A_n$ is a $\Sigma^0_n(X)$ property of $\overrightarrow{p_n}$, and hence is computable in $X^{(n)}$. ∎

It remains to prove the analogue of Lemma 8, namely:

Lemma 15 *There is a set G such that A is $\Sigma^0_n(G)$ and $G^{(n)} \leqslant_T F_n$.*

Proof: As in the case when $n = 1$, we construct G computably in F_n so that every Π^0_n sentence about G is decided by a condition in $\boldsymbol{P}^A_n$. ∎

We are now able to conclude our theorem from Lemmas 14 and 15 by taking $\boldsymbol{c}^{(n)}_\mu = \deg_T(F_n)$.

This completes the proof of Theorem 4. ∎

4 Torsion free abelian groups

As mentioned in the introduction, there is an application of the Main Theorem to effective algebra, answering a question of Downey and Jockusch in [1]. We direct the reader to Downey [1] for any details that do not appear below and for a survey of effective algebra. Recall that a group $(G, \cdot, =)$ is said to have *torsion* if some element has finite order, and *torsion-free* otherwise. If G is isomorphic to a subgroup of $\mathbb{Q}^n$, then the least such n is called the *rank* of the group. The simplest of the torsion-free abelian groups are those of rank 1, that is those that are isomorphic to a subgroup of $(\mathbb{Q}, +, =)$.

Let $p_1, p_2, \ldots$ be the prime numbers in increasing order and let G be a subgroup of $\mathbb{Q}$. The *p-height* $h_p(a)$ of an element $a \in G$ is defined as

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is greatest such that } p^k \mid a \text{ in } G \\ \infty & \text{if } p^k \mid a \text{ for all } k \end{cases}$$

and we define the *characteristic* of a to be the sequence of the p-heights of a,

$$\chi(a) = (h_{p_1}(a), h_{p_2}(a), h_{p_3}(a), \ldots).$$

Notice that since G has rank 1, if $a, b \in G$ with $a, b \neq 0$ then $\chi(a)$ and $\chi(b)$ are equivalent modulo finite differences. We write $\chi(a) =^* \chi(b)$ and define the *type* of $\chi(G)$ to be the $=^*$ equivalence class of $\chi(a)$ for any $a \neq 0$ in G. The classical theorem of Baer is that if G and H have rank 1, then $G \cong H$ if and only if G and H have the same type.

The *standard type of* G, $S(G)$, is defined as

$$S(G) = \{\langle i, j\rangle \mid j \leqslant \text{ the } i\text{th member of } \chi(G)\}$$

for some fixed $a \in G$. If G is X-presented, then $S(G)$ is $\Sigma^0_1(X)$. We say that a group has *finite type* if for all i,

$$|\mathbb{N}^{[i]} \cap S(G)| < \infty.$$

That is, G has no elements of infinite height. In what follows we often identify a group G with a presentation of it.

Recall a group has degree $\boldsymbol{z}$ if $\boldsymbol{z}$ is the least degree in

$$\{\deg(B) \mid B \cong A\}.$$

A group A has 1-degree (jump degree) $\boldsymbol{z}$ if $\boldsymbol{z}$ is the least degree in

$$\{\deg(B)' \mid B \cong A\}.$$

Theorem 16 (Downey, after Knight) *Let* $\boldsymbol{a}$ *be any degree. There exists a torsion free abelian group* G *of rank one and finite type with degree* $\boldsymbol{a}$*.*

Proof: Let $A \in \boldsymbol{a}$. Let G be the rank one group define via the type sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n = 1$ if $n \in A \oplus \overline{A}$ and $a_n = 0$ otherwise. Clearly there is a presentation of G computable in A. Conversely, suppose $H \cong G$. Then H has type sequence $(a_n)_{n\in\mathbb{N}}$. The type sequence of H is $\Sigma^0_1(H)$ and hence $A \oplus \overline{A}$ is $\Sigma^0_1(H)$. Hence $A \leqslant_T H$. ∎

The above notions led Downey and Jockusch to make the following observation.

Theorem 17 (Downey and Jockusch [1]) $\boldsymbol{c}'_\mu(X)$ *exists for all sets* X *if and only if all torsion-free abelian groups of finite type and of rank 1 have* 1-*degree.*

Proof: We present a proof due to Downey and Solomon[1]. We first prove the following set equivalence for G, a torsion free abelian group of rank 1 and finite type,

$$\{\deg_T(H) \mid H \cong G\} = \{\deg_T(Y) \mid S(G) \in \Sigma_1^0(Y)\} \qquad (\dagger)$$

$\subseteq$: If $H \cong G$ then $S(H) =^* S(G)$ and $S(H)$ is $\Sigma_1^0(H)$. Therefore $S(G)$ is $\Sigma_1^0(H)$.

$\supseteq$: Suppose $S(G)$ is $\Sigma_1^0(Y)$. We show that there is a group $H \cong G$ such that $H \equiv_T Y$. We can construct $H \leqslant_T Y$ by using the enumeration of $S(G)$ from Y to add new divisors to H whenever necessary in the following way.

We split the proof into two cases: when $S(G)$ has a finite number of elements of the form $\langle i, j\rangle$ with $j \geqslant 1$ and when it has an infinite number of such elements. The pairs $\langle i, 0\rangle$ are in $S(G)$ for every i, but we are free to ignore these because they add nothing to the structure of the group.

Assume that $S(G)$ has an infinite number of elements of the form $\langle i, j\rangle$ with $j \geqslant 1$. Since $S(G)$ is $\Sigma_1^0(Y)$, there is a 1-1 function $f : \mathbb{N} \to S(G)$ which is Y-computable and enumerates the pairs $\langle i, j\rangle \in S(G)$ with $j \geqslant 1$. Notice that if $f(k) = \langle i, j\rangle$ and $j > 1$, then there is some m such that $f(m) = \langle i, j-1\rangle$.

Let H be the torsion free abelian group with the computable set of generators h and a_k for $k \in \mathbb{N}$ and the following relations. If $f(k) = \langle i, j\rangle$ and $j = 1$, then $p_i \cdot a_k = h$ and if $j > 1$, then $p_i \cdot a_k = a_m$ for m such that $f(m) = \langle i, j-1\rangle$. The intuition is that by induction, we have that $p_i^{j-1} \cdot a_m = h$, which implies that $p_i^j \cdot a_k = h$. Therefore p_i^j divides h and $\langle i, j\rangle$ appears in $S(H)$. Formally, elements of H are finite sums of the form

$$n \cdot h \;+\; \sum_{k \in K} m_k \cdot a_k,$$

where K is a finite set and for each $k \in K$, $1 \leqslant m_k < p_i$ where $f(k) = \langle i, j\rangle$. Addition is done in the obvious way, using the relations above to reduce the

[1]No proof of this result has appeared in print. Solomon noted a flaw in our original proof in an earlier version of this paper.

coefficients in the resulting sum. Because f is Y-computable, it is clear that $H \leqslant_T Y$.

Since G is a torsion free group with rank 1, G is isomorphic to the subgroup of $\mathbb{Q}$ generated by 1 and $1/p_i^j$ for each $\langle i, j\rangle \in S(G)$ with $j \geqslant 1$. We equate G with this subgroup of $\mathbb{Q}$ and define a map $\alpha : H \to G$ by extending $h \mapsto 1$ and $a_k \mapsto 1/p_i^j$ with $f(k) = \langle i, j\rangle$ across H in the natural way. Because α respects the relations in the definition of H, it is a homomorphism and because it maps onto all the generators of G, it is onto. To see that α is 1-1, notice that the generators in G satisfy exactly the same relations as the generators of H. Hence, α is an isomorphism as required.

We are left to consider the case when $S(G)$ contains only finitely many elements of the form $\langle i, j\rangle$ with $j \geqslant 1$. Assume $S(G)$ has m such elements, which we denote by $\langle i_1, j-1\rangle, \ldots, \langle i_m, j_m\rangle$. We define H to be the torsion free abelian group generated by h and a_k for $1 \leqslant k \leqslant m$ and subject to the relations $p_{i_k}^{j_k} \cdot a_k = h$. In this case, H is a computable group and by arguments similar to those in the previous case, H is isomorphic to G.

Assuming G is not the trivial group, then H is infinite. To construct H_0 such that $H_0 \cong H$ with $H_0 \equiv_T Y$ we make an isomorphic copy of H with domain Y. Let $f : \mathrm{dom}(H) \mapsto Y$ be a Y-computable bijection. Define the group structure on Y via f^{-1}. Then $H_0 \cong H$ and $H_0 \equiv_T Y$.

This completes the proof of (†).

Now suppose $\boldsymbol{c}'_\mu(X)$ exists for all sets X. Then there is a least element in the set

$$\{\deg_T(H)' \mid H \cong G\}.$$

Hence G has 1-degree.

Conversely let X be any set. Let $S(G)$ contain

$$\{\langle n, 0\rangle \mid n \geqslant 0\} \cup \{\langle n, 1\rangle \mid n \in X\}.$$

Then for any set Y, X is $\Sigma_1^0(Y)$ if and only if $S(G)$ is $\Sigma_1^0(Y)$. So for any set X there is a set $S(G)$ such that

$$\{\deg_T(Y) \mid X \in \Sigma_1^0(Y)\} = \{\deg_T(Y) \mid S(G) \in \Sigma_1^0(Y)\}.$$

It follows that G having 1-degree implies $\boldsymbol{c}'_\mu(S(G))$ exists, from which $\boldsymbol{c}'_\mu(X)$ must exist. ■

Hence it follows from the Main Theorem that every torsion-free abelian group of finite type and rank 1 has 1-degree.

5 Questions

The Main Theorem raises many natural questions. Analogous questions apply to $\boldsymbol{c}_\mu^{(n)}$.

1. Given a degree $\boldsymbol{a} \geqslant \boldsymbol{0}'$, characterise the sets X for which $\boldsymbol{a} = \boldsymbol{c}'_\mu(X)$.
2. Characterise the sets X for which $X' \in \boldsymbol{c}'_\mu(X)$.
3. Characterise the sets X for which $\boldsymbol{c}'_\mu(X) \in \boldsymbol{0}^{(n)}$ for $n > 0$.
4. Suppose $A \in \boldsymbol{c}'_\mu(X)$. What can be said about the Turing degrees of sets Y for $X \leqslant_T Y \leqslant_T A$?
5. Suppose $A \in \boldsymbol{c}'_\mu(X)$. What can be said about the Turing degrees of sets Y for $A \leqslant_T Y \leqslant_T X'$?
6. Suppose $X_0 \equiv_T X_1$. Under what conditions does
   (a) $\boldsymbol{c}'_\mu(X_0) = \boldsymbol{c}'_\mu(X_1)$?
   (b) $\boldsymbol{c}'_\mu(X_0) < \boldsymbol{c}'_\mu(X_1)$?
   (c) $\boldsymbol{c}'_\mu(X_0) \not\leqslant \boldsymbol{c}'_\mu(X_1)$ and $\boldsymbol{c}'_\mu(X_1) \not\leqslant \boldsymbol{c}'_\mu(X_0)$?
7. A question posed by Jockusch related to 6(a) is the degree-theoretic version of the Main Theorem. Given a degree $\boldsymbol{a}$, does the set
$$\{X' \mid \exists A \in \boldsymbol{a}(A \in \Sigma_1^0(X))\}$$
have least Turing degree?

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