Department of Mathematics, University of California, Berkeley

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2023

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1. Answer six of the nine problems each day. You will get no extra credit for attempting more than 6 problems.
2. The exam lasts 3 hours each day, including time to enter questions in gradescope.
3. Do not answer more than one question on any given piece of paper, as this will confuse the examiners.
4. Submit your answers by uploading pictures or a PDF file to gradescope.
5. The exam is open book: you may use notes or books or calculators or the internet, but may not consult anyone else.
6. In case of questions or unexpected problems during the prelim send email to the chair of the prelim committee at `borcherds@berkeley.edu`. If a correction or announcement is needed during the exam it will be sent as an email to the address you use on gradescope for the prelim, so please keep an eye on this during the prelim.

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Evaluate the infinite product

$$\prod_{n=3}^{\infty} \frac{n^2-4}{n^2-1}$$

Solution: The product of the first k terms is $\frac{1!3!}{0!4!}\frac{k!(k+4)!}{(k+1)!(k+3)!}$, which tends to $1/4$ as k tends to infinity.

Problem 2A. *Score:*

Let $f\colon [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function, and let $g\colon [0,1] \to \mathbb{R}$ be the function

$$g(x) = \min_{0 \le y \le 1} f(x,y)\ .$$

Show in detail that g is continuous on $(0,1)$. (It is also continuous at the endpoints, but don't worry about them.)

Solution: Since $[0,1] \times [0,1]$ is compact, f is uniformly continuous. Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$|f(x_0,y_0) - f(x_1,y_1)| < \epsilon \qquad \text{whenever } |(x_0,y_0) - (x_1,y_1)| < \delta\ .$$

We claim that

$$|g(x_0) - g(x_1)| < \epsilon \qquad \text{whenever } |x_0 - x_1| < \delta\ .$$

Indeed, suppose $|x_0 - x_1| < \delta$. Pick y_0 such that $g(x_0) = f(x_0,y_0)$. Then

$$g(x_1) \le f(x_1,y_0) < f(x_0,y_0) + \epsilon = g(x_0) + \epsilon\ ,$$

and therefore $g(x_1) - g(x_0) < \epsilon$. A similar argument shows that $g(x_0) - g(x_1) < \epsilon$, so $|g(x_0) - g(x_1)| < \epsilon$, and we are done.

(This shows that g is in fact uniformly continuous on $(0,1)$.)

YOUR EXAM NUMBER ___

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Problem 3A. *Score:*

Prove Taylor's theorem with the remainder on the form of Peano: If a real-valued function on the number line has a well-defined nth derivative at $x = 0$, then the error of approximating the function near $x = 0$ by its degree-n Taylor polynomial is $o(x^n)$. (A function f is $o(x^n)$ at a point if f/x^n tends to 0 at this point.) [Do not assume continuity or even existence of the nth derivative in any neighborhood of $x = 0$.]

Solution:

Let f be the function, and $g(x) := f(x)-(f(0)+f'(0)x+f''(0)x^2/2!+\cdots+f^{(n)}(0)x^n/n!)$. To prove that $g(x) = o(x^n)$, compute the limit of $g(x)/x^n$ using L'Hopital's Rule. Namely, since $g(0) = g'(0) = \cdots = g^{(n-1)}(0) = 0$, the limit of the ratio as $x \to 0$ coincides with the limit of $\frac{f^{(n-1)}(x)-f^{(n-1)}(0)}{n!x} - \frac{f^{(n)}(0)}{n!}$ (the expression obtained by differentiating $n-1$ times the numerator and denominator of $g(x)/x^n$), provided that the latter limit exists. By definition of the derivative, the limit of the left fraction equals $f^{(n)}(0)/n!$ which cancels with the right fraction. By L'Hopital's Rule, the limit of $g(x)/x^n$ therefore exists and is equal to 0

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Problem 4A. *Score:*

Suppose that f is a complex polynomial all of whose roots have real part at most 0. Show that if $r > 0$ then $|f(r)| \geq |f(-r)|$. Give an example to show that can be false if the condition that f is a polynomial is replaced by the condition that f is entire.

Solution:

If $f = a(z - b)$ is linear the result is true because $|f(r)|$ is $|a|$ times the distance of r from b, and b is closer to $-r$ than to r as the real part of b is at most 0.

The result follows for any polynomial by writing it as a product of linear polynomials.

The function e^{-z} is entire and has no zeros but does not satisfy the condition $|f(r)| \geq |f(-r)|$

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Problem 5A. *Score:*

Evaluate the contour integral

$$\lim_{R\mapsto+\infty}\int_{c-iR}^{c+iR}\frac{1}{z}dz$$

where c is a nonzero real number. (Warning: the answer depends on c.)

Solution:

The indefinite integral of $1/z$ is a suitable choice of a branch of $\log(z)$, so is given by $\lim R\mapsto\infty\log(c+iR)-\log(c-iR)$. If c is positive this is πi, while if c is negative it is $-\pi i$.

YOUR EXAM NUMBER ___
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Problem 6A. *Score:*

A plane passing through the origin in $\mathbb{R}^3$ intersects the ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$ by an ellipse. Determine how many such sections are circles and find their radiuses.

Solution: By the Cauchy interlacing theorem, the semi-axes $a \geq b > 0$ of the ellipse satisfy $2 \leq b \leq 3 \leq a \leq 4$. Namely, all points of the ellipse $x^2/4 + y^2/9 = 1$ in the plane $z = 0$ are at most distance 3 away from the origin (with the equality held only for the points on the y-axis), and any other plane passing through the origin intersects the plane $z = 0$ in a line, and thus contains such points. This shows that $b \leq 3$ (with the equality achieved only when the plane contains the y-axis). Similarly, $a \geq 3$ follows by intersecting with the plane $x = 0$. Thus, if the section is a circle ($a = b$), then the radius is 3, and the plane must contain the y-axis. Since a plane containing the y-axis is symmetric (together with the ellipsoid) about the plane $y = 0$, the corresponding ellipse has the y-axis as one of its principal ones, and so the other one lies in the plane $y = 0$. Obviously the ellipse $x^2/4 + z^2/16 = 1$ contains two pairs of centrally symmetric points at the distance 3 from the origin, and therefore the ellipsoid has 2 circular sections.

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Problem 7A. *Score:*

Suppose that the square complex matrix A is similar to A^n for some integer $n > 1$. Prove all eigenvalues of A are either zero or roots of unity.

Solution: If a is an eigenvalue then a^n is an eigenvalue of A^n and therefore of A because A and A^n are similar. Similarly $a, a^n, a^{n^2}, \ldots$ are all eigenvalues, so two of these must be equal as the number of eigenvalues is finite. So a is a root of $x^{n^i} = x^{n^j}$ for some distinct integers i, j, whose only roots are 0 and roots of unity.

Problem 8A. *Score:*

Let $\alpha\colon G \to G_1$ and $\beta\colon G \to G_2$ be group homomorphisms.

(a). Show that if $\ker\alpha \subseteq \ker\beta$ and α is surjective (onto) then there is a well-defined group homomorphism $\phi\colon G_1 \to G_2$ such that $\beta = \phi \circ \alpha$.

(b). Show that if $\ker\alpha \not\subseteq \ker\beta$ then there is no such homomorphism ϕ.

Solution: (a). For each $y \in G_1$ there is an $x \in G$ such that $\alpha(x) = y$. We then define $\phi(y) = \beta(x)$. This is well defined because if $\alpha(x') = \alpha(x) = y$, then $\alpha(x'x^{-1}) = \alpha(x)\alpha(x')^{-1} = yy^{-1} = e$, so $x'x^{-1} \in \ker\alpha \subseteq \ker\beta$; hence $\beta(x'x^{-1}) = e$, so $\beta(x')\beta(x)^{-1} = e$, and therefore $\beta(x') = \beta(x)$.

It is a homomorphism because if $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$, then $\alpha(x_1x_2) = y_1y_2$, and thus

$$\phi(y_1y_2) = \beta(x_1x_2) = \beta(x_1)\beta(x_2) = \phi(y_1)\phi(y_2)$$

for all $y_1, y_2 \in G_1$. Finally, ϕ satisfies $\beta = \phi \circ \alpha$ by construction.

(b). We prove the contrapositive.

Assume that ϕ exists. Let $x \in \ker\alpha$. Then $\beta(x) = \phi(\alpha(x)) = \phi(e) = e$, so $x \in \ker\beta$. Thus $\ker\alpha \subseteq \ker\beta$, and we are done.

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Problem 9A. *Score:*

Recall that S_5 and A_5 are the symmetric group and alternating group on 5 letters, respectively.

Prove or give a counterexample: For every $\sigma \in A_5$ there is a $\tau \in S_5$ such that $\tau^2 = \sigma$.

Solution:

Any element σ of odd order $2n+1$ in a group is the square of $\tau = \sigma^{-n}$, so we can assume σ has even order. The only elements $\sigma \in A_5$ of even order are of the forms

$$\sigma = (a\,b)(c\,d)$$

, which is the square of

$$\tau = (a\,c\,b\,d)$$

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Problem 1B. *Score:*

Let $f\colon (a,b] \to \mathbb{R}$ be a function. Assume that f is strictly increasing on (a,b). (This means that $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in (a,b).) Assume also that f is continuous from the left at b. Then show that f is strictly increasing on $(a,b]$.

Solution: It remains only to show that $f(x_1) < f(b)$ for all $x_1 < b$ in $(a,b]$.

Let x_1 be as above, and pick $x' \in (x_1, b)$. Since $f(x) \ge f(x')$ for all $x \in (x', b)$, we have $\lim\limits_{x\to b^-} f(x) \ge f(x')$, and therefore

$$f(x_1) < f(x') \le \lim_{x\to b^-} f(x) = f(b)\ ,$$

as was to be shown.

Problem 2B. *Score:*

By definition, the *unit cube* in the space $C[0,1]$ of continuous real-valued functions on the closed interval $[0,1]$ consists of those functions whose norm $\|f\| := \max_{0\le t\le 1}|f(t)|$ doesn't exceed 1. Find a linear map $\mathbb{R}^3 \to C[0,1]$ such that the inverse image of the unit cube is the unit ball in $\mathbb{R}^3$.

Solution:

Consider a map $\mathbb{R}^3 \to C[0,1]$, $(a,b,c) \mapsto af + bg + ch$, where f, g, h are three continuous functions on $[0,1]$. These functions define a continuous parametric curve in $\mathbb{R}^3 : x = f(t), y = g(t), z = h(t)$. Our goal is to find a parametric curve, such that it lies in the "layer" $-1 \le ax + by + cz \le 1$ exactly when $a^2 + b^2 + c^2 \le 1$. For this, it is sufficient that the convex hull of the parametric curve is exactly the unit ball $x^2 + y^2 + z^2 \le 1$. To construct such a curve, take a *Peano map*, i.e. a continuous mapping of the interval $[0,1]$ *surjectively* onto the unit square $[0,1] \times [0,1]$, and compose it with any continuous map wrapping the square (e.g. cylindrically) around the unit sphere $x^2 + y^2 + z^2 = 1$.

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Problem 3B. *Score:*

Show that the recursive sequence $x_{n+1} = x_n/2+1/x_n$ with the initial value $x_0 = 1.5$ converges to $\sqrt{2}$, and that x_{10} has at least 1000 correct decimal digits.

Solution: The fixed point of $x/2 + 1/x$ is $x = \sqrt{2}$ indeed. Moreover, due to the inequality between the arithmetic and geometric means, $(x + 2/x)/2 \geq \sqrt{x \cdot 2/x} = \sqrt{2}$, i.e. the fixed point is also a critical point of the function. Therefore, if $x_n = \sqrt{2}+\epsilon_n$ where the (necessarily positive) error $\epsilon_n < 10^{-k}$, the next error ϵ_{n+1} will be of the order $\epsilon_n^2 < 10^{-2k}$. More precisely, from the geometric series expansion:

$$\frac{1}{\sqrt{2}+\epsilon_n} = \frac{1}{\sqrt{2}} - \frac{\epsilon_n}{2} + \frac{\epsilon_n^2}{2(\sqrt{2}+\epsilon_n)},$$

and therefore

$$\epsilon_{n+1} = \frac{\sqrt{2}+\epsilon_n}{2} + \frac{1}{\sqrt{2}+\epsilon_n} - \sqrt{2} = \frac{\epsilon_n^2}{2(\sqrt{2}+\epsilon_n)}.$$

Since $\epsilon_0 < 1.5 - 1.41 < 1/10$, it follows that $\epsilon_{10} < 1/10^{2^{10}} = 1/10^{1024}$.

Problem 4B. *Score:*

Show that there is a function holomorphic on the open unit disc that is a bijection from the open unit disc to the vertical strip $0 < \Re z < 1$, where $\Re z$ is the real part of z. You may not use the Riemann mapping theorem.

Solution: We give such a function as a composition of several functions as follows.

(1) Shift the unit disc by 1 so that the boundary passes through 0.

(2) Apply $1/z$ so that the image is a half plane

(3) Apply a linear transformation to make the half plane the half plane with positive real part.

(4) Apply the log function to get the strip with imaginary part between $\pm\pi i/2$

(5) Apply a linear transformation to make this the strip with real part between 0 and 1.

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Problem 5B. *Score:*

Find the radius of convergence of the Taylor series of $1/(e^x + e^{-x})$ at the point $x = 1$.

Solution: The radius of convergence is the distance from 1 to the nearest singularity. The singularities are at the points where $e^{2x} = -1$, so $2x = \pm i\pi, \pm 3i\pi, \ldots$. The nearest singularities to 1 are therefore $\pm i\pi/2$ so the radius of convergence is $\sqrt{1 + \pi^2/4}$.

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Problem 6B. *Score:*

Prove that if two real square matrices are similar by conjugation by a complex matrix, then they are similar by conjugation by a real matrix.

Solution: This follows from the real version of the Jordan canonical form theorem, but can be also derived directly. Namely, suppose $B = CAC^{-1}$ where $C = D + iE$ is an invertible complex matrix and A, B, D, E are real. Then $CA = BC$ and therefore $DA = BD$ and $EA = BE$. Even if neither D nor E is invertible, the polynomial $\det(D + tE)$ cannot be identically zero (since $\det(D + iE) \neq 0$), and so it should be non-zero for some real values of t. For such values, $B = (D + tE)A(D + tE)^{-1}$.

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Problem 7B. *Score:*

Given an example of two square complex matrices that have the same minimal polynomial and the same characteristic polynomial but are not similar.

Solution:

Take two 4 by 4 matrices in Jordan normal form with all eigenvalues 0, such that the first has Jordan blocks of size 2, 2 and the second has Jordan blocks of size 2, 1, 1.

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Problem 8B. *Score:*

Let n be a positive integer, and p any prime. Prove that over a finite field F of $p^{\phi(n)}$ elements the polynomial $x^n - 1$ factors into linear factors. (Here ϕ is Euler's totient function.)

Solution:

Let $n = p^r m$ where m is coprime to p. Over F, we have $x^n - 1 = (x^m - 1)^{p^r}$ (since $\binom{p}{k} \equiv 0 \mod p$ for $k = 1, \ldots, p-1$). Since the multiplicative group of a finite field is cyclic, to show that $x^m - 1$ has m distinct roots in F, it suffices to check that the order $|F^\times| = p^{\phi(n)} - 1$ is divisible by m. But since p and m are coprime, we have $\phi(n) = \phi(p^r)\phi(m)$, and $p^{\phi(n)} \equiv (p^{\phi(m)})^{\phi(p^r)} \equiv 1 \mod m$ by Euler's theorem.

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Problem 9B. *Score:*

Give an example of a commutative ring which has an infinite descending chain of distinct prime ideals $I_1 \supset \cdots \supset I_n \supset \cdots$ (Recall that an ideal I of a commutative ring R is called prime if R/I is an integral domain.)

Solution:

In the ring R of (say, complex coefficient) polynomials $\mathbb{C}[x_1, x_2, \ldots, x_n, \ldots]$ in infinitely many variables, take I_n to be the ideal generated by all x_i with $i > n$. The quotient R/I_n is isomorphic to the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ which has no zero divisors, implying that the ideal I_n is prime.