**Math H185**
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## Final Examination

Open book, open notes. The points for each question are in parentheses.

1. (15) (a) For which positive numbers $a$ will at least one value of $i^a$ be real?
   (b) For which positive $a$ will all values of $i^a$ be real?

2. (15) Find the images under the linear-fractional transformation $\varphi(z) = \dfrac{z-i}{z+i}$ of the right half-plane $\mathrm{Re}\ z > 0$, the left half-plane $\mathrm{Re}\ z < 0$, and the sector $\dfrac{\pi}{4} < \mathrm{Arg}\ z < \dfrac{3\pi}{4}$.

3. (10) Let the power series $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R$, where $0 < R < \infty$. Let the power series $\sum_{n=0}^{\infty} b_n z^n$ have radius of convergence $\infty$. Prove the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ has radius of convergence $\infty$.

4. (15) Evaluate
$$\int_{|z|=3\pi} \frac{z^n}{e^z - 1} dz, \qquad n = 0, 1, 2, \ldots,$$
   where the circle $|z| = 3\pi$ has the counterclockwise orientation.

5. (15) Let $f$ be a complex-valued harmonic function in a domain $G$ such that $f^2$ is also harmonic. Prove either $f$ is holomorphic or $\bar{f}$ is holomorphic. (Suggestion: The complex differential operators $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$ could be useful here.)

6. (15) The entire function $f$ is said to be of exponential type if there are positive constants $C$ and $k$ such that $|f(z)| \le Ce^{k|z|}$ for all $z$. Prove that if $f$ is of exponential type then $f'$ is also of exponential type.

7. (15) Let the function $f$ be holomorphic in the strip $-1 < \mathrm{Im}\ z < 1$, real on the real axis, and of positive imaginary part in the strip $0 < \mathrm{Im}\ z < 1$.
   (a) Prove $f$ has negative imaginary part in the strip $-1 < \mathrm{Im}\ z < 0$.
   (b) Prove $f'(x) \ge 0$ for $x$ real.
   (c) Prove $f'(x) > 0$ for $x$ real.

8. (20) Evaluate
$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx.$$
   Justify each step.

9. (10) Let $f$ be an entire function such that the set $\mathbb{R} \cap f^{-1}(\mathbb{R})$ has a finite limit point. Prove $f(x)$ is real for all real $x$.