

*Spr 2004*      MATH 128A FINAL EXAM      *Prof Zworski*

Please try to answer each question on the page on which it is presented,  
including the back of the page. Generous partial credit will be given.

Good luck!

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Problem 7	
<b>Total</b>	

**Problem 1. (20 points)**

a) Find the inverse of the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -2 & 2 & -3 \\ 4 & 5 & 5 \end{pmatrix}.$$

b) Write a formula for the condition number,  $\text{cond}(A)$ , of an  $n \times n$  matrix  $A$ .

c) Using the  $\|\cdot\|_\infty$  norm find the condition number of the matrix in part a). You can quote the fact that if  $A$  is an  $n \times n$  matrix then

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|.$$

d) Prove that if  $Ax = b \neq 0$ , and  $A(x + \delta x) = b + \delta b$  then

$$\frac{\|\delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\delta b\|}{\|b\|}.$$

**Problem 2. (15 points)**

- a) What is the  $LU$  decomposition of a matrix  $A$ ? Are  $L$  and  $U$  unique? What is the condition on  $A$  guaranteeing existence of  $L$  and  $U$ ?
- b) Find the  $LU$  decomposition of the following matrix:

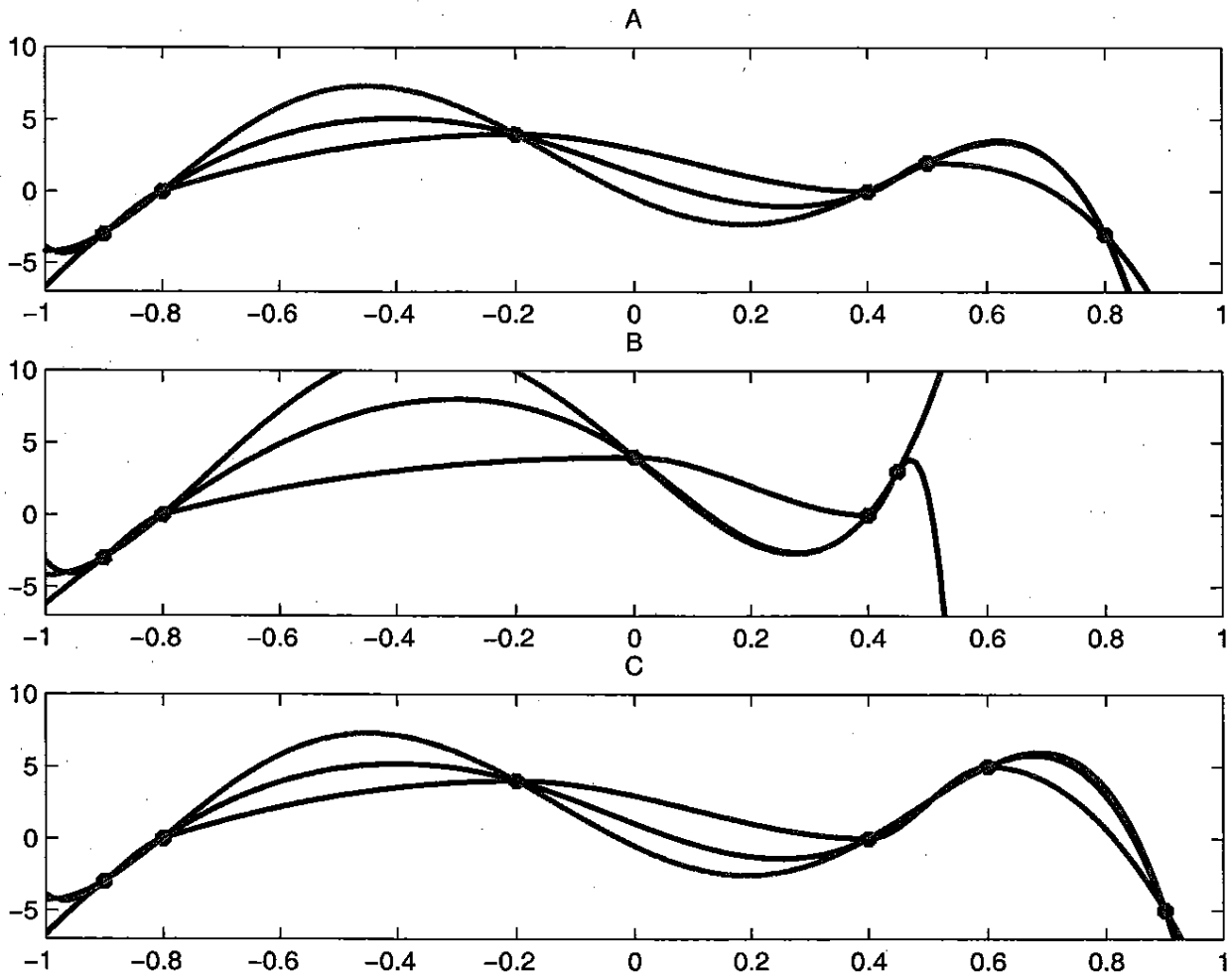
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

- c) How many *multiplications* are needed to compute the  $LU$  decomposition of a general  $n \times n$  matrix?

**Partial credit will be given – try to get the right order of magnitude!**

**Problem 3. (10 points)**

- a) Define a spline (specifying the free or clamped boundary conditions).
- b) In the following three pictures three different sets of data are interpolated using Lagrange interpolation, splines, and piecewise cubic Hermite interpolation. Determine which color corresponds to which interpolation (credit will be given for motivating your answer).



**Problem 4. (10 points)**

The following matlab code produces a row vector `mystery` from three row vectors `x`, `y`, and `xx` (where `x` and `y` are row vectors of the same size).

What are we computing here and what particular method are we using? The more you can motivate your answer the better.

```
function mystery=mystery(x,y,xx)
[P,N]=size(x);
[R,M]=size(xx);
Q=zeros(N);
Q(:,1)=y';
w=zeros(N);
for i=1:N
for k=1:i
w(i,k)=x(i)-x(i-k+1);
end
end
for j=1:M
for i=2:N
for k=2:i
Q(i,k)=(xx(j)-x(i-k+1))*Q(i,k-1)-(xx(j)-x(i))*Q(i-1,k-1)/w(i,k);
end
end
y(j)=Q(N,N);
end
mystery=y;
```

**Problem 5. (10 points)**

- a) Define the order of convergence of a sequence of numbers,  $x_n$ .
- b) What is the order of convergence of the sequence generated by Newton's method?
- c) Consider the polynomial  $P(z) = z^2 + 1$ . Use Newton's method with two iterations, deflation, and initial condition  $z_0 = 1 + i$ , to approximate both zeros of  $P(z)$ . Compute the absolute and relative errors of your solutions.

**Problem 6. (15 points)**

Let  $y'(t) = f(t, y(t))$  be a first order ordinary differential equation.

a) State as clearly as you can the condition on  $f$  which guarantees the existence and uniqueness of solution to that equation with initial condition  $y(0) = 1$ . Is that condition satisfied for  $f(t, y) = \cos(\pi(t + y))$ ?

b) Use Euler's method with  $h = 0.25$  to approximate the solution of

$$y' = \cos(\pi(t + y)), \quad y(0) = 1,$$

on the interval  $[0, 1]$ .

c) Derive Taylor's method of order 2 for the same equation – you do not have to provide the numerical solution.

**Problem 7. (20 points)**

- a) State Simpson's rule for  $\int_0^1 g(x)dx$  and  $n = 2$ . Assuming the degree of accuracy of Simpson's method *derive* the *precise* error bound.
- b) State the rule given by the three point Gaussian quadrature for the *same* integral.
- c) Use Simpson's rule of part a) to approximate the length of the graph of  $\cos(\pi x)/\pi$  between  $x = 0$  and  $x = 1$  (you can quote the fact that if  $f(x)$  is a function defined for  $a \leq x \leq b$  then the arclength of the graph of  $f$  is given by  $\int_a^b (1 + f'(x)^2)^{1/2} dx$ ).
- d) Use the rule from part b) to do the same thing. Find the absolute and relative errors in case c) and d), knowing that the exact length is  $1.2160\dots$  (working with 5 significant digits).