SPRING 2003 D. Geba

FINAL EXAM — MATH 104

① Define the sequence $(x_n)_n$, by $x_n = \int_0^n \frac{dx}{x^2+3x+2}$.

Compute $x_n$ and then find $\lim_n x_n$.

② Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = (x-a)\cdot|x-b|$, where $a, b \in \mathbb{R}$. Prove that $f$ is differentiable on $\mathbb{R} \iff$ $\iff a = b$, and in that case compute the derivative

③ Let $f: (0,\infty) \to \mathbb{R}$, $f(x) = \begin{cases} \frac{\ln x}{x-1}, & \text{for } x \neq 1 \\ 1, & \text{for } x = 1. \end{cases}$

Prove that $f$ is differentiable and its derivative is continuous.

④ Let $f, g: (a,b) \to \mathbb{R}$ two not equal continuous functions. Prove that $h: (a,b) \to \mathbb{R}$, defined by:

$$h(x) = \begin{cases} f(x), & x \in \mathbb{Q} \cap (a,b) \\ g(x), & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (a,b) \end{cases}$$

does not admit any antiderivative.

⑤ Prove that for $f: [0,1] \to \mathbb{R}$, differentiable, with the derivative continuous and $f(1) = 0$, the following inequality holds:

$$\int_0^1 (f'(x))^2 dx \geq 3\left(\int_0^1 f(x)dx\right)^2$$