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Spring 2000

Math 104-2: Midterm II (Take home)

Due 4/7/2000 in class: late submissions not accepted.

The Rules:

- You may use books, notes, and any other reference materials.
- You may work together with other students in the class, but you may not consult anyone else.
- Every solution you submit must be fundamentally your own: even if you discuss the main ideas with a classmate, each of you should write it up separately.

In many ways, this exam is just like a normal problem set. However, because it counts for a larger percentage of your grade, and you have more time to do it, my standards for solutions will be somewhat higher. I strongly recommend that you first write a rough draft of your solution to each problem, then go over it to make sure that:

- The steps in your proof are arranged in a systematic, logical progression.
- The connection between each step and the preceding one is clear.
- Each step is suitably justified. (In particular, solutions that contain only algebraic manipulations and no words are almost never OK.)
- All possible special cases are considered.
- You have done everything the problem asks for.

When you write your final draft, think of it almost as a mini-essay. I realize that in doing your regular homework, you are usually pressed for time and have a million other things to do – so I don't expect the proofs to be quite as polished. But in this midterm, I want you to make the extra effort!

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Please complete this part of the page, and hand it in with your solutions.

For each problem, please list any sources you used besides Rudin and your notes, and the names of any classmates with whom you discussed it:

Problem 1.

Problem 2.

Problem 3.

Problem 4.

Problem 5.

Your signature below confirms that you did not consult any other people or sources besides those indicated; and that the solutions to all the problems are your own work.

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(please sign here)

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1. A very energetic kangaroo is tied to a post by an infinitely stretchable rubber band. On the post sits a flea, desperate to get to the kangaroo. But before the flea can do anything, the kangaroo jumps away, landing 100 yards from the post. The flea jumps too, landing on the rubber band, 1 yard from the post. The kangaroo jumps 100 yards again, stretching the rubber band: so he is now 200 yards away from the post, and the flea is 2 yards away. Then the flea again jumps 1 yard further on the rubber band. If the kangaroo and the flea keep taking turns this way, does the flea ever catch up to the kangaroo? (Justify your answer! A mere "yes" or "no" will count for nothing.)

2. Consider a sequence of numbers $\{c_n\}$ given by the recursive formula
$$c_{n+1} = c_n^{c_n}.$$
Analyze the limit behavior of this sequence for different values of $c_1 > 0$. (Justify all the steps in your calculations!)

3. Let $\{a_n\}$ be a sequence in a metric space X, and suppose that, for each $k \geq 2$, the subsequence
$$a_k, a_{2k}, a_{3k}, \ldots$$
converges.
   (a) Prove that all these subsequences converge to the *same* limit.
   (b) Does $\{a_n\}$ itself necessarily converge?

4. Suppose that $\sum a_n$ converges, with each $a_n > 0$, and let
$$r_n = \sum_{k=n}^{\infty} a_k,$$
for each $n \in \mathbb{N}$. Show that
   (a) the sequence $\{r_n\}$ is decreasing, with limit 0.
   (b) $\sum a_n r_n$ converges.
   (c) $\sum \frac{a_n}{r_n}$ diverges. [*Hint:* Show that the sequence of partial sums for this series is not Cauchy.]

5. Consider a sequence of complex numbers $\{a_n\}$.
   (a) If $\sum(a_n + a_{n+1})$ converges, does $\sum a_n$ necessarily converge?
   (b) If $\sum(|a_n| + |a_{n+1}|)$ converges, does $\sum a_n$ necessarily converge?

   If your answer is "yes", provide a proof; if your answer is "no", provide examples of both divergence and convergence!

Good luck and have a good spring break!