MATH 110-1 Linear Algebra
Spring 2000 I. Novik

Midterm II

1. **(30 pts)** Supply short proofs for the following statements. Each part is worth **6 pts**

   (a) If V is a finite-dimensional vector space, then its dual space V^* has the same dimension as V.

   (b) If A is an $n \times n$ matrix satisfying $A^5 = 0$, then $\det(A) = 0$.

   (c) If matrix B is similar to matrix C, then B^2 is similar to C^2.

   (d) If $T : V \to V$ is a linear transformation on a finite-dimensional vector space V and λ is an eigenvalue of T, then $\lambda^3 - 3\lambda^2 + 7$ is an eigenvalue of $T^3 - 3T^2 + 7I$.

   (e) Let $T : V \to V$ be a linear operator on an inner product space V. Show that if $< T(u), v >= 0$ for every $u, v \in V$ then $T = 0$.

2. **(15 pts)** What are the algebraic multiplicities of the eigenvalues of the differentiation transformation D ($D(P) = P'$) on the space $P_3(\mathbf{R})$ of polynomials of degree ≤ 3? What are the dimensions of eigenspaces? Is D diagonalizable?

3. **(15 pts)** Let V be a vector space over $\mathbf{R}$, and let $T : V \longrightarrow \mathbf{R}^n$ be a **one-to-one** linear transformation. Show that for $u, v \in V$ the formula

$$< u, v >_V := < T(u), T(v) >$$

defines an inner product on V. (Here $< -, - >$ is the standard inner product on $\mathbf{R}^n$).

4. **(20 pts)**

(a) Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A.
(Hint: Find an eigenvector)

(b) Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A.

5. **(20 pts)** Let $T \in L(V, V)$, and let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. Prove that $f(T)$ is a zero map, where

$$f(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n).$$

Hint: what is $(f(T))(v_i)$? (also see Problem 1d)