Math 115 — Professor K. A. Ribet

Final Exam — December 14, 1999

This is a closed-book exam: no notes, books or calculators are allowed. Explain your answers in complete English sentences. No credit will be given for a "correct answer" that is not explained fully.

Each question is worth 6 points.

1. Let n be an integer greater than 1. Let p be the smallest prime factor of n. Show that there are integers a and b so that $an + b(p-1) = 1$.

2. Using the identity $27^2 - 8 \cdot 91 = 1$, describe the set of all integers x that satisfy the two congruences $x \equiv \begin{cases} 35 & \bmod 91 \\ 18 & \bmod 27 \end{cases}$.

3. Let $m = 2^2 3^3 5^5 7^7 11^{11}$. Find the number of solutions to $x^2 \equiv x \bmod m$.

4. Calculate $\left(\dfrac{-30}{p}\right)$, where p is the prime 101. Justify each equality that you use.

5. Write $2 + \sqrt{8}$ as an infinite simple continued fraction.

6. Find the number of primitive roots mod p^2 when p is the prime 257.

7. Express the continued fraction $\langle 6, 6, 6, \ldots \rangle$ in the form $a + b\sqrt{d}$, with a and b rational numbers and d a positive non-square integer.

8. Suppose that $p = a^2 + b^2$, where p is an odd prime number and a is odd. Show that $\left(\dfrac{a}{p}\right) = +1$. (Use the Jacobi symbol.)

9. Let n be an integer. Show that n is a difference of two squares (i.e., $n = x^2 - y^2$ for some $x, y \in \mathbf{Z}$) if and only if n is either odd or divisible by 4.

10. Let n be an integer greater than 1. Prove that 2^n is not congruent to 1 mod n.