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70 Evans Hall

Fall 1995, Math 250A

Final Exam

Saturday, 16 October 1995
12:30-3:30 PM

1. (7 points) Prove that every finitely generated group G is a homomorphic image of a free group F on finitely many generators.

2. (7 points) Show that if $f: A \to B$ is a homomorphism of rings, and P is a prime ideal of B, then the ideal $f^{-1}(P) = \{a \in A \mid f(a) \in P\}$ of A is prime.

3. (7 points) Let R be a commutative ring, and suppose that for some positive integer n, the element $n1 \in R$ is not a unit in R. Show that there exists a homomorphism of R onto a field of nonzero characteristic.

4. (10 points) Let R be a ring, let r be an element of R, and for any left R-module M, let $\mathrm{Ann}_M(r)$ denote the abelian group $\{x \in M \mid rx = 0\} \subseteq M$ ("the annihilator of r in M"). Clearly, an R-module homomorphism $f: M \to N$ carries $\mathrm{Ann}_M(r)$ into $\mathrm{Ann}_N(r)$, and this restriction of f is a homomorphism of abelian groups. Let us call this restricted homomorphism $\bar{f}: \mathrm{Ann}_M(r) \to \mathrm{Ann}_N(r)$.

Suppose now that

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C$$

is an exact sequence of left R-modules. Prove that the sequence

$$0 \to \mathrm{Ann}_A(r) \xrightarrow{\bar{a}} \mathrm{Ann}_B(r) \xrightarrow{\bar{b}} \mathrm{Ann}_C(r)$$

of abelian groups is also exact.

5. (10 points) Let R be a unique factorization domain which is not a field, and k its field of fractions. Show that k is not algebraically closed.

6. (20 points) Let K/k be a finite Galois extension, let $G = G(K/k)$, and let S be any transitive nonempty G-set. We will also regard K as a G-set (considering the action of G on K by automorphisms as an action on the set K).

(a) (8 points) Show that there exists an element $\alpha \in K$ such that the orbit of α under the action of G on K is isomorphic as a G-set to S.

(b) (4 points) For α as in (a), express $[k(\alpha) : k]$ in terms of the properties of the G-set S, justifying your answer.

(c) (8 points) For α as in (a), state a condition on the G-set S which is necessary and sufficient for the normal closure of $k(\alpha)$ to be the whole field K, and prove this equivalence.

7. (28 points) (a) (14 points) Prove the following result (Hilbert's Theorem 90, multiplicative form). You may assume any results proved *before* it in Lang.

> Let K/k be a cyclic Galois extension of degree n with Galois group G, and σ a cyclic generator of G. Let $\beta \in K$. Then $N_k^K(\beta) = 1$ if and only if there exists $\alpha \neq 0$ in K such that $\beta = \alpha/\sigma\alpha$.

(b) (14 points) Deduce from the result of (a) that if k is a field, n an integer not divisible by the characteristic of k such that k contains a primitive nth root of unity ζ_n, and K a cyclic extension of k of degree n, then K is the splitting field over k of a polynomial of the form $X^n - a$ $(a \in k)$ which is irreducible over k.

(Note: this is a result from Lang, but with some details changed. Namely, you are *not* asked to prove the converse, as Lang does, but you *are* asked to prove that the polynomial is irreducible over k, which Lang doesn't, and to show that K is the splitting field of that polynomial, rather than the result of adjoining a single root. Again, you may assume results proved earlier in Lang, but not the version of *this* result that he proves.)

8. (11 points) Let K be an extension of a field k, let Γ be a subset of K, and S an algebraically independent subset of Γ. Lang notes (without proof) that if $k(\Gamma) = K$, then there is a transcendence basis for K over k containing S and contained in Γ. Prove (without using that result of Lang's) that this conclusion is in fact true under the weaker assumption that K is *algebraic* over $k(\Gamma)$ (not necessarily equal to it).

You will not need to use anything Lang proves about transcendence bases; just the definition (a transcendence basis is a maximal subset of K algebraically independent over k) and general results about algebraicity. You may take for granted that the union of a chain of algebraically independent subsets of K is algebraically independent.