Final Exam - Linear Algebra

Math 110, December 15th, 2008.
Instructor: Christian Zickert

The exam consists of 6 problems that are worth a total of 200 points. **No electronic equipments are allowed.** You have 3 hours to complete the exam. Good luck!

YOUR NAME:____

Problem 1 (35 points). Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

(a) **(7 points).** Show that the characteristic polynomial of A splits and that -1 is the only eigenvalue.

(b) **(8 points).** Show that $N(A+I)$ is generated by $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and that $N\left((A+I)^2\right)$ is generated by $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Problem continues on the next page.

(c) **(10 points).** Compute the Jordan canonical form of A.

(d) **(10 points).** Find a Jordan canonical basis for A.

Go to the next page for problem 2.

Problem 2 (30 points). This problem consists of 3 unrelated problems that are worth 10 points each.

(a) Let V be a complex vector space of dimension 4, and let T be a linear operator on V with 2 as the only eigenvalue. Determine all possible Jordan canonical forms of T.

(b) Let F be a field and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F)$. Show that the characteristic polynomial f_A of A is given by

$$f_A(t) = t^2 - (a+d)t + ad - bc,$$

and use this to prove that $A^2 - \text{tr}(A)A + \det(A)I = 0$.

(c) Give an example of a linear operator on a 4-dimensional real vector space with no eigenvalues.

Go to the next page for problem 3.

Problem 3 (40 points). Let x be a non-zero complex number and let

$$A = \begin{pmatrix} x + x^{-1} & -1 \\ 1 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{C}).$$

(a) **(8 points).** Determine the eigenvalues of A.

(b) **(8 points).** Show that A is diagonalizable if and only if $x \neq \pm 1$.

(c) **(8 points).** For $x \neq \pm 1$, determine a basis β for $\mathbb{C}^2$ such that $[L_A]_\beta$ is diagonal, and find a matrix $C \in M_{2\times 2}(\mathbb{C})$ such that $C^{-1}AC = [L_A]_\beta$.

Problem continues on the next page.

(d) **(8 points).** For $x = -1$ and $x = 1$, determine the Jordan canonical form for A and a Jordan canonical basis for A.

(e) **(8 points).** Show that A is normal if and only if x is purely imaginary, i.e., if and only if $x = ai$ for some $a \in \mathbb{R}\backslash\{0\}$.

Go to the next page for problem 4.

Problem 4 (30 points). Let V be an n-dimensional vector space, and let U and Z be subspaces of dimension k and l. Let $U \cap Z$ denote the intersection of U and Z.

(a) **(15 points).** Show that $\dim(U \cap Z) \geq k + l - n$.

Hint: Let $U \times Z$ denote the vector space $\{(u, z) \mid u \in U, z \in Z\}$ and consider the nullspace of the linear map $T\colon U \times Z \to V$ given by sending (u, z) to $u - z$.

(b) **(15 points).** Show that any two 3-dimensional subspaces of $P_4(\mathbb{R})$ contain a common subspace of dimension 1.

Go to the next page for problem 5.

Problem 5 (30 points). Let $v = (1, 0, 2, -1) \in \mathbb{R}^4$ and let l be the line spanned by v.

(a) **(15 points).** Determine an orthonormal basis for the orthogonal complement $l^\perp$ of l.

(b) **(15 points).** Let $x = (2, 1, 3, 2)$. Find $u \in l$ and $z \in l^\perp$ such that $x = u + z$.

Go to the next page for problem 6.

Problem 6 (35 points). Let V be a finite dimensional inner product space, and let $T\colon V \to V$ be a linear operator. Suppose v_1 and v_2 are vectors in V satisfying

$$\|v_1\| = 1, \ \|v_1 + v_2\| = 1 \text{ and } \langle v_1, v_1 + v_2\rangle = 0.$$

Let $W = \mathrm{span}(v_1, v_2)$ and let $\beta = \{v_1, v_2\}$ and $\beta' = \{v_1, v_1 + v_2\}$.

(a) **(8 points).** Show that W is 2-dimensional and that β and β' are bases for W.

(b) **(10 points).** For the rest of the problem, suppose W is T-invariant and that

$$[T_W]_\beta = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $[T_W]_{\beta'} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

Problem continues on the next page.

(c) **(10 points).** Show that $[(T_W)^*]_\beta = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$.

(d) **(7 points).** Show that $x = v_1 + v_2$ is an eigenvector of $(T_W)^*$ with eigenvalue 1.

End of exam.