MATH 104 FINAL

May 17, 2003 70 Evans H. Wu

Your Name: ____________________

1. (30 pts.) Find the radius of convergence and determine the exact interval of convergence of each of the following three series. (a) $\sum_n \left(\frac{5^n}{n!}\right) x^n$

(b) $\sum_n \left(\frac{2^n}{\sqrt{n}}\right) x^{2n+3}$

(c) $\sum_{n} \left(\frac{2 + (-1)^n}{3} \right) x^n$

2. (10 pts.) $\frac{d}{dx} \int_0^{x^2} e^{\sin u} \, du$

3. (20 pts.) Let f be a continuous function defined on $[a, b]$ and let F be the function defined on $[a, b]$ by

$$F(x) = \int_a^x f \ .$$

Prove that F is differentiable in (a, b) and that $F'(x_0) = f(x_0)$ for every $x_0 \in (a, b)$.

4. (10 pts.) Let $\{s_n\}$ be a sequence such that $\lim_{n\to\infty} s_{2k} = L$ and $\lim_{n\to\infty} s_{2k+1} = L$. Does $\{s_n\}$ converge, and why?

5. (15 pts.) Prove that if $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$.

6. (15 pts.) Given power series $\sum_n a_n x^n$. If the series converges for at least one nonzero x, prove that there is a number R, $0 \leq R \leq \infty$, so that the power series converges for all $x \in (-R, R)$ and (in case $R < \infty$) diverges for all x so that $|x| > R$.

7. (32 pts.) (a) (7 pts.) State the ϵ-δ definition of *uniform continuity* for a function defined on $S \subset \mathbb{R}$.

(b) (15 pts.) Prove that the function $f : [1, \infty) \to \mathbb{R}$, such that $f(x) = 1/x^2$, is uniformly continuous on $[1, \infty)$ by directly verifying the ϵ-δ definition.

(c) (10 pts.) Give another proof of (b).

8. (33 pts.) (a) (8 pts.) State the definition of *uniform convergence* of a sequence of functions g_n defined on $S \subset \mathbb{R}$ to a function g which is also defined on S.

(b) (15 pts.) Does the sequence $f_n(x) = \dfrac{nx}{1 + 10\,nx}$ converge uniformly to $\frac{1}{10}$ on $(0, 1]$? Explain.

(c) (10 pts.) Does the sequence f_n in (b) converge uniformly to $\frac{1}{10}$ on $[\frac{1}{15}, \infty)$? Explain.

(9) (15 pts.) (a) (7 pts.) Define the *continuity* of a function f at a point $x_0 \in \mathbb{R}$.

(b) (8 pts.) Let F be continuous at 1 and let $F(1) > 0$. Prove that there is an $\epsilon > 0$ so that $F(x) > 0$ for all $x \in (1 - \epsilon, 1 + \epsilon)$.

(10) (20 pts.) Let f, g be continuous functions defined on $[0,2]$ so that $f(0) = -1.5$ and $g(0) = 3$. Assume $f' \geq 1$ and $g' \leq -2$ on $(0,2)$. Prove that $f(x_0) = g(x_0)$ for some $x_0 \in [0,2]$.