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Fall 2002, Math 250A
Final Exam

19 December, 2002
12:30-3:30 PM

1. (28 points; 7 points each.) Complete the following definitions.

(a) If $\mathscr{A}$ and $\mathscr{B}$ are categories, then a *functor* $F: \mathscr{A} \to \mathscr{B}$ means ... (Note: Don't worry about problems in wording arising from the question of whether the objects and morphisms in a category form a set. A definition that would be correct in a category where they do form a set will be accepted.)

(b) A group G is said to be *solvable* if ...

(c) An algebraic extension K of a field k is said to be *normal* over k if it satisfies three equivalent conditions, *one* of which is ...

(d) If $k \subseteq K$ are fields, then a *transcendence basis* for K over k means ...

2. (32 points; 8 points each.) For each of the items listed below, either *give an example,* or give a brief reason why *no example exists.* (If you give an example, you do *not* have to prove that it has the property stated. Examples should be specific for full credit; i.e., even if there are many objects of a given sort, you should name one.)

(a) A non-Noetherian commutative ring.

(b) A nonzero commutative ring R such that there exists no homomorphism from R onto a field.

(c) A nontrivial automorphism σ of the field $\mathbf{Q}$ of rational numbers. (''Nontrivial'' meaning ''not equal to the identity automorphism''.)

(d) A finite field containing a primitive 5th root of unity.

3. (40 points, 5 points per step) In proving each statement below, you may assume all the earlier statements in this problem, even if you did not succeed in proving them all. Since I don't expect everyone to get every part of this question (though I would be very happy if you did), and since later steps depend on earlier ones, it is important to keep this in mind!

You can also, of course, call on any results proved in our readings in Lang and the Companion.

Let k be a perfect field containing a square root of -1. (If you are shaky on the concept of perfect field, you may, for nearly full credit, replace the condition that k be perfect by the condition that it have characteristic 0, but you must show where you use that assumption.) Let k^{a} be an algebraic closure of k, and α an element of $k^{\mathrm{a}} - k$.

(a) Show that there exists a subfield K of k^{a} containing k and maximal for the property $\alpha \notin K$; i.e., such that $\alpha \notin K$, but there is no subfield of k^{a} strictly larger than K which does not contain α. (A detailed proof of this would involve a large number of trivial verifications. *Don't* show those verifications; instead, say what straightforward set of facts needs to be verified, and how one deduces from these that such a K exists.)

(b) Show that there exists an automorphism σ of k^{a} over the subfield K of part (a) such that $\sigma(\alpha) \neq \alpha$.

☛ Note: Throughout the remaining parts, K and σ will have the meanings given them in parts (a) and (b) above. By an *intermediate field* we will mean a field E with $K \subseteq E \subsetneq k^{\mathrm{a}}$. An intermediate field E will be called *nontrivial* if it is distinct from K. ☚

(c) Show that every nontrivial intermediate field contains α. Deduce from this that $K(\alpha)$ is the least nontrivial intermediate field, and that the fixed field of σ is K.

(d) Show that $[k^{\mathrm{a}} : K]$ is infinite. Deduce from this that σ has infinite order.

(e) Show that if E is an intermediate field that is finite and Galois over K, then its Galois group $\mathrm{Gal}(E/K)$ is generated by the restriction $\bar{\sigma}$ of σ to E. (Begin by pointing out why $\bar{\sigma}$ will take E into itself.) Deduce from this that every intermediate field F that is finite over K is Galois over K.

(f) Show that if E is a nontrivial intermediate field finite over K, then $\mathrm{Gal}(E/K)$ has a greatest proper subgroup (a proper subgroup containing all other proper subgroups). Deduce from this that $\mathrm{Gal}(E/K)$ (which we have seen is the finite cyclic group $\langle\bar{\sigma}\rangle$) has prime power order, p^m, where $p = [K(\alpha) : K]$ and $m > 0$. (Hint for the second sentence: Recall a description of all subgroups of a finite cyclic group Z_n, and the inclusion relations among them.)

(g) Show that the set of intermediate fields which are finite over K is totally ordered by inclusion.

(h) Show that the chain of intermediate fields which are finite over K has no largest element, and deduce that for every positive integer n it contains a field of degree p^n over K.

Final remark: Every intermediate field is algebraic over K, hence is a union of intermediate fields finite over K. Since these form a chain, there is only one intermediate field *infinite* over K, the full union of that chain. So the set of all intermediate fields looks like $K = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_m \subsetneq \ldots \subsetneq E_\infty = k^{\mathrm{a}}$, with one field E_m of degree p^m for each nonnegative m, and one field k^{a} of infinite degree over K. (Nothing for you to prove here; just a summary of the consequences of the above statements.)