F02

FINAL EXAM - MAT 110, SECTION 6 D. Geba

① Let V a finite-dimensional vector space over a field F and $\beta = \{v_1, v_2, \dots, v_n\}$ an ordered basis for V.

Consider also Q an $n \times n$ invertible matrix with elements from F and define

$$w_i = \sum_{j=1}^{n} q_{ji} v_j \quad \text{for every } i \in \{1, 2, \dots, n\}.$$

Prove that $\gamma = \{w_1, w_2, \dots, w_n\}$ is also a basis for V.

② Let A, B two $n \times n$ matrices with real elements, such that $A^3 = B^5 = I_n$ and $AB = BA$. Prove that $A+B$ is invertible.

③ Let V an inner product space and $S = \{v_1, \dots, v_n\}$ an orthonormal subset of V. Prove that for every $x \in V$, the following inequality holds:

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

Prove also that we have equality in the previous estimate if and only if $x \in$ span S.

④ Let $T: V \longrightarrow V$ a diagonalizable linear operator and consider $P_T(t)$ its characteristic polynomial.

a) Prove that $P_T(t) = (-1)^n (t - \lambda_1) \cdots (t - \lambda_n)$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of T.

b) Prove that if we define the operator
$P_T(T) = (-1)^n (T-\lambda_1 I)(T-\lambda_2 I) \cdots (T-\lambda_n I)$,
then $P_T(T) = T_0$ - zero operator.
(*Hint*: Find a basis $\beta = \{v_1, \ldots, v_n\}$ for V such that $P_T(T)(v_i) \neq 0$, $\forall\, i \in \{1, 2, \ldots, n\}$).