Math 113: Introduction to Abstract Algebra

Final exam, May 21st, 2002

Weingart

Name: ______________________________

Signature: ___________________________

There are 10 problems on this final worth 20 points each, however you should not work on more than 9 problems of your choice dropping the last one. In any case you will only get credit for 9 of the 10 problems. Successful final!

1	2	3	4	5	6	7	8	9	10	Total

Problem 1: (20 points)
Find all group homomorphisms $\phi : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_9$. Which of these group homomorphisms are actually ring homomorphisms?

Problem 2: (20 points)
Formulate the Chinese Remainder Theorem and the Theorems of Lagrange, Euler and Kronecker.

Problem 3: (20 points)
How many different solutions do you expect to find for the equation $x^2 - x = 0$ for $x \in \mathbb{Z}_{14}$?
Factorize $x^{10} + \overline{10} \in \mathbb{Z}_{11}[x]$ into irreducible polynomials.

Problem 4: (20 points)
Solve the two congruences $2x \equiv 17 \mod 7$ and $5x \equiv 3 \mod 11$ simultaneously for $x \in \mathbb{Z}$.

Problem 5: (20 points)
Let R be a ring. Its center is defined to be the set of all $a \in R$ commuting with every $b \in R$

$$Z(R) := \{ a \in R \mid ab = ba \text{ for all } b \in R \}.$$

e. g. $Z(\mathbb{H}) = \mathbb{R}$. Show that $Z(R)$ is a subring of R but no ideal in general. Moreover if R is a ring with unity and $a \in Z(R) \cap R^*$ is a unit of R in its center, then $a^{-1} \in Z(R)$ and consequently $Z(R)^* = Z(R) \cap R^*$.

Problem 6: (20 points)
Recall that the radical $\sqrt{I} \supset I$ of an ideal I in a commutative ring R is defined to be the ideal

$$\sqrt{I} := \{ a \in R \mid a^n \in I \text{ for some } n \geq 1 \}$$

of all elements a of R such that some power a^n, $n \geq 1$, of a is in I. An ideal I is called radical if it agrees with its radical $I = \sqrt{I}$. Show that the ideal $n\mathbb{Z} \subset \mathbb{Z}$ is radical if and only if n is square free. You may want to use the Fundamental Theorem of Arithmetic.

Problem 7: (20 points)
Show that there are matrices $K, L \in M_2(\mathbb{Q})$ with

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = K \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L$$

and conclude that the only ideal I of $M_2(\mathbb{Q})$ with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in I$ is $M_2(\mathbb{Q})$ itself.

Problem 8: (20 points)
The polynomial $x^3 - 6x^2 + x - 1$ is irreducible over $\mathbb{Q}$ as it is of degree ≤ 3 and has no zero in $\mathbb{Q}$. Define addition and multiplication on $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ such that the resulting ring is isomorphic to $\mathbb{Q}[x]/(x^3 - 6x^2 + x - 1)$. Is this ring an integral domain?

Problem 9: (20 points)
Show that for an ideal $\overline{J} \subset \mathbb{Z}_n$ the set $J := \{ a \in \mathbb{Z} \mid \overline{a} \in \overline{J} \}$ is an ideal of $\mathbb{Z}$ containing $n\mathbb{Z} \subset J \subset \mathbb{Z}$. Describe all ideals of $\mathbb{Z}_n$.

Problem 10: (20 points)
Consider an idempotent x with $x^2 = x$ in a ring R with unity 1 and show that $1 - x$ is an idempotent as well. If R is in addition commutative we can consider the two ideals $(x) = \{ xk \mid k \in R \}$ and $(1 - x) = \{ (1 - x)k \mid k \in R \}$ generated by x and $1 - x$ as rings in their own right. Show that the map

$$\phi : \ R \longrightarrow (x) \times (1 - x), \quad a \longmapsto (xa, (1 - x)a)$$

is a ring homomorphism (guess an inverse ring homomorphism to show it is an isomorphism).