The Zeta Function of a Graph [*]

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Abstract

Original "*Many Cheerful Facts*" Abstract:
When studying a finite graph, it is often useful to consider its closed paths. A particularly interesting kind of closed path is one with no immediate back-tracking, which we will call a *prime geodesic*. These "primes" play a role analogous to the usual primes of arithmetic. In particular, the problem of counting prime geodesics of certain lengths, turns out to be very similar to that of the classical *Prime Number Theorem*.

Through Ihara's analogue of the Riemann zeta function, we will explore these new primes. We will also mention *Ramanujan graphs* which are connected to the graph-theoretic "Riemann hypothesis".

Finally, time permitting, we will recast everything in a geometric point of view. Replacing our nice discrete graph with something bigger, like a Riemann surface of constant negative curvature.

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[*] These are expanded lecture notes from a talk entitled *The Prime Number Theorem for Graphs*, given on April 6th 2005 in the Berkeley math department's *Many Cheerful Facts* seminar. They were created to convince myself of the validity of the Prime Geodesic Theorem for graphs. As such, they are not polished; and there may exist many irregularities. It should also be stated that none of the major results presented here are original. V1.1 Questions and comments can be sent to: **zywina@math.berkeley.edu**

1 Motivation: the Riemann zeta function

1.1 Basics:

In number theory we are interested in studying the properties of the primes

$$\mathcal{P} = \{2, 3, 5, 7, 11, 13, \cdots\}.$$

In this pursuit, Euler was among the first to consider the function

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s},$$

where s is a real number (From freshman calculus; the series converges for $s > 1$, and diverges for $s \leq 1$). The connection between ζ and the prime numbers is seen through the *Euler product*:

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}},$$

which also converges for all $s > 1$.

Writing the right-hand side as $\prod_{p\in\mathcal{P}}(1+p^{-s}+(p^2)^{-s}+(p^3)^{-s}+\cdots)$, we see that the Euler product is just an analytic expression of the *fundamental theorem of arithmetic.*

The limit $\lim_{s\to 1^+}\zeta(s)=\infty$ shows that there must be infinitely many factors in the Euler product, i.e. there are infinitely many prime numbers. The zeta function is used to not only show there are infinitely many primes, but to try to describe their distribution.

1.2 Analytic continuation and the Prime Number Theorem

Riemann considered $\zeta(s)$ as a function of a complex variable. The series expression of $\zeta(s)$ converges absolutely for $\Re s>1$, and hence to a holomorphic function there. Riemann was able to show that it analytically continues to a (necessarily unique) meromorphic function on all of $\mathbb{C}$. We will henceforth view ζ as a function on all of $\mathbb{C}$. $\zeta(s)$ turns out to be holomorphic everywhere except $s=1$, where it has a simple pole.

Riemann's study of the zeta function was motivated by the *Prime Number Theorem* (which was only a conjecture of Gauss and Legendre at the time)
Consider the important counting function,

$$\pi(x)=\#\{p\in\mathcal{P}|p<x\}.$$

Theorem 1.1. *(Prime Number Theorem)*

$$\pi(x)\sim\frac{x}{\log x}\ \textit{as}\ x\to\infty$$

This means that $\lim_{x\to\infty}\pi(x)/(\frac{x}{\log x})=1.$ $\square$

Remark 1. *Rewriting this as*

$$\frac{\pi(x)}{x}\sim\frac{1}{\log x},$$

gives an asymptotic expression for the density of primes. Gauss conjectured the PNT as a boy (in fact he conjectured a better approximation).

Hadamard and de le Valee Poussin (independently) where able to show that $\zeta(s)$ is nonzero when $\Re s=1$, and from this were able to deduce the P.N.T (in fact the PNT is equivalent to this!). In general, the better the knowledge of the zeros of ζ, the better the knowledge of the distribution of primes. Which leads to the the famous
Conjecture 1.2. *(Riemann Conjecture) If* $\zeta(s)=0$ *where* $0<\Re s<1$*, then* $\Re s=1/2$*. That is, all "non-trivial zeros" of* ζ *lie on the "critical line".*

Outside the"critical strip" $0<\Re s<1$, the zeros of $\zeta(s)$ are simply $-2,-4,-6,\cdots$ (the "trivial zeros" of ζ).

Remark 2. *The Riemann hypothesis for $\zeta(s)$ is equivalent to the following:*

$$\pi(x) = li(x) + O(x^{1/2+\epsilon}),$$

where $li(x) = \int_2^x \frac{dt}{\log t}$.

The idea behind this talk is to carry these ideas, and apply them to graphs. Zeta functions are analytic expressions containing information about "primes", and through analysis we hope that they will give up their secrets.

2 A brief review of graph theory

We will keep this informal. A graph X consists of a vertex set $V = V(X)$, an edge set $E = E(X)$, and a map which associates to each edge, two vertices in V (the "endpoints" of the edge). The edges of our graphs will be undirected.

All of the graphs we will consider will be finite (ie. $\#V, \#E < \infty$). We write $v \sim w$, to mean that the vertices v and w are connected by an edge. [1]

A graph is *k-regular* if every vertex is an endpoint of exactly k edges.

A *walk* of length r from x to y, is a sequence $x = v_0, v_1, \cdots, v_r = y \in V$ such that $v_i \sim v_{i+1}$ for $i = 0, \cdots, r-1$. A graph is *connected* if any two vertices have a walk between them.

We will be especially interested in *proper walks*, which are the walks with no back-tracking (ie. We never have $v_i = v_{i+2}$).

A *closed geodesic* is a closed path (ie. same initial and final point), such that there is no backtracking if we go around twice. (Equivalently, it is a closed proper walk with the initial and final edges different)

If γ is a closed geodesic, we denote by γ^r the closed geodesic obtained by repeating the walk γ r times.

We say two closed geodesics $(x_0, \cdots, x_n)$ and $(y_0, \cdots, y_m)$ are *equivalent* if $m = n$ and there is a d such that $y_i = x_{i+d}$ for all i (where the subscripts are interpreted modulo n).

Definition 2.1. A closed geodesic which is not the power of another is called a *prime geodesic*. An equivalence class of a prime geodesic is called a *prime geodesic class* (or just a *prime*).

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[1] We will implicitly assume that each edge has distinct end-points, and at most one edge connects any two vertices. Including the other cases does not make the following that much harder.

Given a path γ we denote by $l(\gamma)$ its length (for simplicity, we assume each edge has length 1). The length of a prime $\mathfrak{p}$, denoted by $\ell(\mathfrak{p})$, is the length of any representative.

Number the vertices, $V(X) = \{1, \cdots, n\}$. The *adjacency matrix* is the $n \times n$-matrix, $A = A(X)$, such that

$$A_{i,j} = \text{ the number of edges connecting } i \text{ and } j.$$

Note that everything we do with A will be independent of the initial numbering.

Definition 2.2. (Technical definition) Define

$$g(X) := gcd\{\ell(\mathfrak{p}) : \mathfrak{p} \in \mathcal{P}\}.$$

Remark 3. *In fact for a k-regular graph ($k > 2$), $g(X) = 1$ or $g(X) = 2$ (at least I think so). It is recommended at first reading that you assume that $g(X) = 1$ (equivalently non-bipartite). This g plays a bigger role when you deal with non-regular graphs.*

3 Ihara zeta function

3.1 Definitions

From now on, X will be a finite connected k-regular graph, where $k > 2$. Let $q = k - 1$.

We now, in analogy with the first section, define the set of *primes* of X to be

$$\mathcal{P} = \{\mathfrak{p} | \ \mathfrak{p} \text{ a prime of } X\}$$

We need to associate to each prime some notion of size [2]. To accomplish this we define the *norm* map,

$$\begin{array}{rcl} \mathcal{N} : \mathcal{P} & \rightarrow & \mathbb{Z}^{+} \\ \mathfrak{p} & \mapsto & q^{\ell(\mathfrak{p})} \end{array}$$

We now define the *zeta function* associated to X to be

$$\zeta_X(s) = \prod_{\mathfrak{p} \in \mathcal{P}} \frac{1}{1 - \mathcal{N}(\mathfrak{p})^{-s}}.$$

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[2] This was not necessary for the Riemann zeta funtion, because a prime already has a natural size, $\mathcal{N}(p) = p$.

We will see later that the product converges for all $\Re s > 1$.

Rewriting we get

$$\zeta_X(s) = \prod_{\mathfrak{p}\in\mathcal{P}} \frac{1}{1 - q^{-s\ell(\mathfrak{p})}}.$$

Which suggests considering the following power series,

$$Z_X(T) = \prod_{\mathfrak{p}\in\mathcal{P}} \frac{1}{1 - T^{\ell(\mathfrak{p})}} \in \mathbb{Z}[[T]].$$

Note: $\zeta_X(s) = Z_X(q^{-s})$.

Theorem 3.1. *(Ihara)* $Z_X(T) \in \mathbb{Z}[[T]]$ *is a* rational function *of* T.

In particular,

$$Z_X(T) = \frac{1}{(1-T^2)^{\chi} \cdot \det(I - AT + qT^2 I)},$$

where $\chi := (q-1)|V|/2$ *(the Euler characteristic of* X*),*

and A *is the adjacency matrix of* X*.*

Proof. Omitted. Can be done combinatorial. □

Proposition 3.2. *The product defining* $\zeta_X(s)$ *is convergent for* $\Re s > 1$.
$\zeta_X(s)$ *extends to a meromorphic function on* $\mathbb{C}$*, with no zeros.*

Proof. The first part is a consequence of $Z_X(T)$ having radius of convergence $1/q$ (see Theorem 3.6). The second part is a consequence of Theorem 3.1. □

3.2 Eigenvalues

Proposition 3.3. *Let* A *be the adjacency matrix of* X*.* $\lambda = k$ *is an eigenvalue of* A*, with multiplicity 1.*

Proof. Let $x = (1, 1, \cdots, 1)^t$. Using that X is k-regular note that $Ax = kx$.
Now let $v = (x_1, \cdots, x_n)^t$ be any eigenvector of A with eigenvalue k. Without loss of generality suppose $|x_1| = \max_i |x_i|$ and $x_1 > 0$. Then

$$kx_1 = \sum_{j=1}^{n} A_{1,j} x_j \leq \sum_{j=1}^{n} A_{1,j} x_1 = kx_1.$$

So when $A_{1,j} \neq 0$, we have $x_j = x_1$. In particular, $x_j = x_1$ for all j adjacent to 1. Repeating the process with these new vertices, and using that X is connected we find that $x_1 = \cdots = x_n$. So the eigenvectors corresponding to k are generated by $(1, 1, \cdots, 1)^t$. $\square$

Proposition 3.4. *Let A be the adjacency matrix of X. $\lambda = -k$ is an eigenvalue of A if and only if X is bipartite. If X is bipartite, then $-k$ is an eigenvalue with multiplicity 1.*

Proof. Same spirit as last proof. Refer to a graph theory book. $\square$

Lemma 3.5. *Let A be the adjacency matrix of X. If λ is an eigenvalue of A, then $|\lambda| \leq k$.*

Proof. There is a $v = (x_1, \cdots, x_n)^t \neq 0$ such that $Av = \lambda v$. Without loss of generality assume that $|x_1| = \max_i |x_i|$

$$|\lambda||x_1| = \Big|\sum_{j=1}^{n} A_{1,j}x_j\Big| \leq |x_1| \sum_{j=1}^{n} A_{1,j} = |x_1| \cdot k$$

Where the last equality uses that X is k-regular. $\square$

Remark 4. *All of the eigenvalues of A are real, since A is symmetric.*

Theorem 3.6. *The* series *$Z_X(T)$ has radius of convergence $1/q$. When analytically continued to $\mathbb{C}$, $Z_X(T)$ is holomorphic on the circle $|T| = 1/q$, except for a simple pole at $1/q$ (and if X is bipartite another simple pole at $-1/q$).*

Proof. Consider the denominator of the rational expression of $Z_X(T)$:

$$(1 - T^2)^\chi \cdot \det(I - AT + qT^2 I) = (1 - T)^\chi (1 + T)^\chi \prod_i (1 - \lambda_i T + qT^2),$$

where λ_i are the eigenvalues of A. We need to find the zeros that are closest to the origin.

Since $1 - k(1/q) + q(1/q)^2 = 0$ we know that $1/q$ is a zero. We claim that there are none closer.

Let z be a zero of the polynomial such that $|z| \leq 1/q$. There is an eigenvalue λ of A such that

$$1 - \lambda z + qz^2 = 0.$$

Hence,

$$|z| = \Big|\frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q}\Big| \leq 1/q$$

or

$$\left|\lambda \pm \sqrt{\lambda^2 - 4q}\right| \leq 2.$$

If $\lambda^2 - 4q \leq 0$, then

$$4q = \lambda^2 + (4q - \lambda^2) \leq 4,$$

which is impossible. Therefore, we may assume $\lambda^2 - 4q > 0$.

Suppose $\lambda > 0$. Then $\left|\lambda \pm \sqrt{\lambda^2 - 4q}\right| \leq 2$ implies that $0 < \lambda - \sqrt{\lambda^2 - 4q} \leq 2$.

$$\begin{array}{rcl} \lambda & \leq & 2 + \sqrt{\lambda^2 - 4q} \\ \lambda^2 & \leq & 4 + 4\sqrt{\lambda^2 - 4q} + (\lambda^2 - 4q) \\ q - 1 & \leq & \sqrt{\lambda^2 - 4q} \\ q^2 - 2q + 1 & \leq & \lambda^2 - 4q \\ (q+1)^2 & \leq & \lambda^2 \\ k & \leq & \lambda \end{array}$$

By Lemma 3.5, $\lambda = k$. We find that $z = 1/q$ or $\frac{2k-2}{2q} = 1$. That we have a simple pole at $1/q$ follows from Prop 3.3.
Similarily, for $\lambda \leq 0$, we find $\lambda = -k$. The rest follows from Prop 3.4.

□

Theorem 3.7. *Let $z \in \mathbb{C}$ be a zero of the polynomial $Z_X(T)^{-1}$, then*

$$1/q \leq |z| \leq 1.$$

Also, all of the zeros on $|z| = 1$ come from the factor $(1 - T^2)^{\chi}$ and one more at 1 from the eigenvalue $\lambda = k$ term (if X is bipartite, there is another zero at -1 from the eigenvalue $\lambda = -k$).

Proof. Theorem 3.6 shows that $1/q \leq |z|$. We still need to show that $|z| \leq 1$.

We may assume by Theorem 3.1 that z is a root of the polynomial $\det(I - AT + qT^2 I)$. There is an eigenvalue λ of A such that

$$1 - \lambda T + qT^2 = q(T - z)(T - z_2).$$

Using $zz_2 = 1/q$ and $|z_2| \geq 1/q$ (Theorem 3.6), we find that $|z| \leq 1$. This also shows that if $|z| = 1$ then $|z_2| = 1/q$; the theorem now follows from Theorem 3.6.

□

Remark 5. *The map* $\mathbb{C} \to \mathbb{C} : s \mapsto T := q^{-s}$ *takes the line* $\{s : \Re s = 1\}$ *to the circle* $\{T : |T| = 1/q\}$. *Theorem 3.6 and 3.1 gives us that* $\zeta_X(s)$ *has no zeros and the location of poles on* $\Re s = 1$. *This is the key information needed for the proof of the Prime Number Theorem in the classical case (and also below).*

Remark 6. *Similarily, the previous theorem implies that* $\zeta_X(s)$ *has no poles (or zeros) outside the strip* $0 \leq \Re s \leq 1$.

3.3 $Z_X(T)$ as a generating function

We can also view $Z_X(T)$ as a generator function containing combinatorial information. (To people who have dealt with zeta functions of curves, etc.., this should look familiar).

Proposition 3.8.

$$Z_X(T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} T^m\right),$$

where N_m *is the number of closed geodesics of length* m.

Proof.

$$\begin{aligned} \log Z_X(T) &= \sum_{\mathfrak{p}\in\mathcal{P}} -\log(1 - T^{\ell(\mathfrak{p})}) \\ &= \sum_{\mathfrak{p}\in\mathcal{P}} \sum_{n=1}^{\infty} T^{\ell(\mathfrak{p})n}/n \end{aligned}$$

$$\begin{aligned} T \cdot Z_X'(T)/Z_X(T) &= \sum_{\mathfrak{p}\in\mathcal{P}} \sum_{n=1}^{\infty} \ell(\mathfrak{p}) T^{\ell(\mathfrak{p})n} \\ &= \sum_{m=1}^{\infty} \left(\sum_{\substack{\mathfrak{p}\in\mathcal{P} \\ \ell(\mathfrak{p})|m}} \ell(\mathfrak{p}) \right) T^m \\ &= \sum_{m=1}^{\infty} \left(\sum_{\substack{p \text{ prime geodesic} \\ \ell(p)|m}} 1 \right) T^m \\ &= \sum_{m=1}^{\infty} N_m T^m \end{aligned}$$

We can now deduce,

$$Z_X(T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} T^m\right).$$

□

4 The Prime Geodesic Theorem

4.1 First guess

Fix a finite connected k-regular graph X (for now suppose $g(X) = 1$). As before let $q = k - 1$. In analog with the arithmetic case define

$$\pi_X(x) = \#\{\mathfrak{p} \in \mathcal{P} | \mathcal{N}(\mathfrak{p}) < x\}.$$

Guess 1.

$$\pi_X(x) \sim \frac{x}{\log x} \textit{ as } x \to \infty$$

□?

$$\begin{aligned} \pi_X(x) = \#\{\mathfrak{p} \in \mathcal{P} | q^{\ell(\mathfrak{p})} < x\} &= \#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < \log_q x\} \\ \implies \#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < y\} &= \pi_X(q^y). \end{aligned}$$

Therefore, we have:
Guess 2. *(Prime Geodesic Theorem?)* [3]

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < y\} \sim \frac{q^y}{y \log q}$$

as $y \to \infty$.

4.2 Towards the Prime Geodesic Theorem

We will first consider X a non-bipartite graph. Consider T times the logarithmic derivative of $Z_X(T)$:

$$f(T) := TZ'_X(T)/Z_X(T) = \sum_{n \geq 1} N_n T^n.$$

Proposition 4.1. $N_n \sim q^n$ *as* $n \to \infty$.

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[3] See Theorem 4.7 for the correct version.

Proof. By Theorem 3.1, we know that $f(T)$ has meromorphic extension to the entire complex plane. By Theorem 3.6, we know that the only pole or zero of $Z_X(T)$ in the disk $|T| \leq 1/q$ is the simple pole at $1/q$.
From complex analysis we find that (for sufficiently small $\epsilon > 0$)

$$\begin{aligned}
I_n &:= \frac{1}{2\pi i}\int_{|T|=1/q+\epsilon} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \\
&= \frac{1}{2\pi i}\int_{|T|=1/q+\epsilon} \frac{1}{\zeta^n}\frac{Z'_X(\zeta)}{Z_X(\zeta)} d\zeta \\
&= Res_{\zeta=1/q}\left(\frac{1}{\zeta^n}\frac{Z'_X(\zeta)}{Z_X(\zeta)}\right) + Res_{\zeta=0}\left(\frac{f(\zeta)}{\zeta^{n+1}}\right) \\
&= -q^n + N_n
\end{aligned}$$

$$\begin{aligned}
I_n &= O(1/(1/q+\epsilon)^{n+1})) \\
&= O(1/(1/q+\epsilon)^n) \\
&= O(q^n/(1+q\epsilon)^n) \\
&= o(q^n)
\end{aligned}$$

The result follows. □

We can now prove the following, called the Prime Geodesic Theorem in the literature.
Theorem 4.2.

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) = n\} \sim q^n/n$$

as $n \to \infty$. [4]

Proof. By Proposition 8.1,

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) = n\} = \frac{1}{n}\sum_{d|n} \mu(n/d) N_d.$$

This shows that (using the last proposition)

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) = n\} = N_n/n + O(\frac{1}{n} n q^{n/2}).$$

This and $N_n \sim q^n$ gives us the theorem. □

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[4] If X was bipartite, the results of this section could still be carried out. The big difference being that $Z_X(T)$ has an extra pole at $-1/q$. We would then derive

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) = 2n\} \sim q^{2 \cdot n}/n,$$

where $n \to \infty$.

4.3 Conclusion of the proof of the prime geodesic theorem

It might help at first reading to assume that $g := g(X) = 1$.

We have the following fact from calculus:
Lemma 4.3. *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *be sequences such that*

$$\begin{aligned} &(i) && \text{For } n \gg 0,\ b_n > 0, \\ &(ii) && \lim_{n\to\infty} \frac{a_n}{b_n} = c, \\ &(iii) && \lim_{m\to\infty} \sum_{n=1}^{m} b_n = \infty. \end{aligned}$$

Then we have

$$\lim_{m\to\infty} \frac{\sum_{n=1}^{m} a_n}{\sum_{n=1}^{m} b_n} = c.$$

□

Proposition 4.4.

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < g \cdot y\} \sim \sum_{n<y} \frac{q^{g\cdot n}}{n},$$

as $y \to \infty$.

Proof. Let $b_n = \#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) = g \cdot n\}$. and $a_n = q^{g\cdot n}/n$. Then by Theorem 4.2, Lemma 4.3(i) and (ii) holds with $c = 1$. We have

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < g \cdot y\} = \sum_{n<y} b_n.$$

It is now easy to see that Lemma 4.3 (iii) holds.
From Lemma 4.3, we conclude that

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < g \cdot y\} \sim \sum_{n<y} q^{g\cdot n}/n,$$

as $m \to \infty$.

□

Lemma 4.5. *For* $\lambda > 1$,

$$\int_n^{n+1} \frac{\lambda^t}{t} dt \sim \frac{\lambda - 1}{\log \lambda} \frac{\lambda^n}{n},$$

as $n \to \infty$.

Proof.

$$\begin{aligned}\int_n^{n+1} \frac{\lambda^t}{t} dt &= \frac{1}{\log \lambda} \frac{\lambda^t}{t}\Big|_n^{n+1} + \frac{1}{\log \lambda} \int_n^{n+1} \lambda^t \frac{dt}{t^2} && \text{(Integration by parts.)} \\ &\sim \frac{1}{\log \lambda} \left(\frac{\lambda^{n+1}}{n+1} - \frac{\lambda^n}{n} \right) \\ &= \frac{\lambda^n}{\log \lambda} \left(\frac{\lambda}{n+1} - \frac{1}{n} \right) \\ &= \frac{\lambda^n}{\log \lambda} \frac{\lambda n - (n+1)}{n(n+1)} \\ &\sim \frac{\lambda^n}{\log \lambda} \frac{\lambda - 1}{n}\end{aligned}$$

□

Lemma 4.6. *For $\lambda > 1$,*

$$\int_1^x \frac{\lambda^t}{t} dt \sim \frac{\lambda^x}{x \cdot \log \lambda},$$

as $x \to \infty$.

Proof.

$$\begin{aligned}\int_1^x \frac{\lambda^t}{t} dt &= \frac{1}{\log \lambda} \frac{\lambda^t}{t}\Big|_1^x + \frac{1}{\log \lambda} \int_1^x \lambda^t \frac{dt}{t^2} && \text{(Integration by parts.)} \\ &\sim \frac{\lambda^x}{x \cdot \log \lambda}\end{aligned}$$

□

We know from Proposition 4.4 that

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < g \cdot y\} \sim \sum_{n<y} \frac{q^{g \cdot n}}{n}.$$

So let $b_n = \frac{q^{g \cdot n}}{n}$ ($b > 0$ and $\sum_{k=1}^{\infty} b_n = \infty$). Let $a_n = \frac{\log \lambda}{\lambda - 1} \int_n^{n+1} \frac{\lambda^t}{t} dt$, where $\lambda = q^g$. By Lemma 4.5, $\lim_{n \to \infty} a_n / b_n = 1$. We can now again apply Lemma 4.3 to see that

$$\sum_{n<y} \frac{q^{g \cdot n}}{n} \sim \sum_{n<y} \frac{\log \lambda}{\lambda - 1} \int_n^{n+1} \frac{\lambda^t}{t} dt = \frac{\log \lambda}{\lambda - 1} \int_1^y \frac{\lambda^t}{t} dt,$$

as $\mathbb{N} \ni y \to \infty$. So we can write (using Lemma 4.6),

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < g \cdot y\} \sim \frac{1}{\lambda - 1} \frac{\lambda^y}{y},$$

as $\mathbb{N} \ni y \to \infty$.

We have proven:

Theorem 4.7. ***(Prime Geodesic Theorem)***

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < g \cdot y\} \sim \frac{1}{q^g - 1} \frac{q^{gy}}{y},$$

as $\mathbb{N} \ni y \to \infty$. $\square$

5 The Riemann hypothesis

Let A be the adjacency matrix of X.
Definition 5.1. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A. Define

$$\lambda(X) = \max_{|\lambda_i| < k} |\lambda_i|.$$

Definition 5.2. We say a graph X is *Ramanujan* if

$$\lambda(X) \leq 2\sqrt{k-1}.$$

Ramanujan graphs have played a role in current research in network theory and communications. They are good examples of "expanders".
Explicit constructions of families of Ramanujan graphs have been constructed by often deep results from number theory and algebraic geometry.

Theorem 5.3. *X is Ramanujan, if and only if, $\zeta_X(s)$ satisfies the Riemann hypothesis; i.e. If $\zeta_X(s) = \infty$ and $0 < \Re s < 1$, then $\Re s = 1/2$.*

Proof. The Riemann hypothesis for ζ_X is equivalent to the following Riemann hypothesis for Z_X: If z is a root of $Z_X(T)^{-1}$ such that $q^{-1} < |z| < 1$, then $|z| = q^{-1/2}$.

First suppose $Z_X(T)$ satisfies the Riemann hypothesis.
Let λ be an eigenvalue of A, $\lambda \neq \pm k$.
Choose a $z \in \mathbb{C}$ such that $1 - \lambda z + qz^2 = 0$, then z is a zero of $Z_X(T)^{-1}$. By Theorems 3.7 and 3.6 we find that $q^{-1} < |z| < 1$. So by the assumed R.H., $|z| = q^{-1/2}$.

$$\lambda = \frac{1 + qz^2}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z} + q|z|^2 z}{|z|^2} = \frac{\overline{z} + qq^{-1}z}{q^{-1}} = (\overline{z} + z)q = 2\Re(z) \cdot q$$

Which gives us

$$|\lambda| \leq 2|z| \cdot q \leq 2q^{-1/2}q = 2\sqrt{q} = 2\sqrt{k-1}$$

as desired.
Therefore, X is Ramanujan.

Conversely suppose that X is Ramanujan.
Let z be a zero of $Z_X(T)^{-1}$ such that $q^{-1} < |z| < 1$. By Theorems 3.7 and 3.6, we know there is an eigenvalue $\lambda \neq \pm k$ of A such that $1 - \lambda z + qz^2 = 0$. Then

$$z = \frac{\lambda \pm \sqrt{\lambda^2 - 4q}}{2q}.$$

By hypothesis $|\lambda| \leq 2\sqrt{k-1}$, so $\lambda^2 - 4q \leq 0$.

• If $\lambda^2 - 4q < 0$, then z is *not* real.
However,

$$\lambda = \frac{1 + qz^2}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z} + q|z|^2 z}{|z|^2}$$

is real. So we must have $q|z|^2 = 1$, i.e. $|z| = q^{-1/2}$.
• Otherwise if $\lambda^2 - 4q = 0$, then

$$z = \lambda/(2q) = \pm 2\sqrt{q}/(2q) = \pm q^{-1/2}.$$

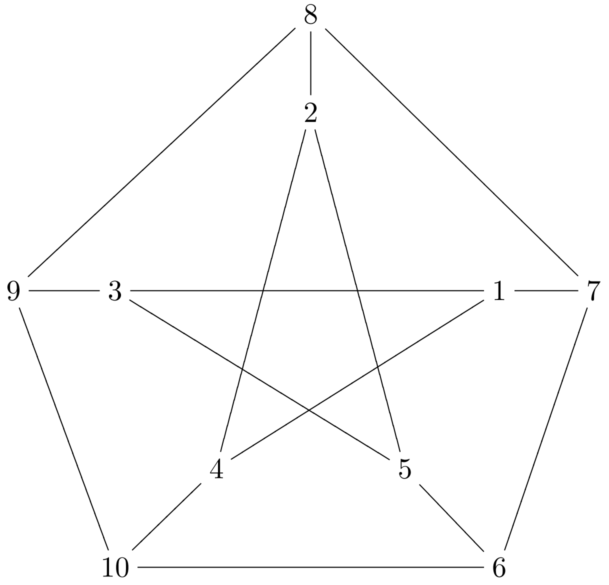
We have verified the Riemann hypothesis. □

6 An example: The Petersen Graph

6.1 The zeta function

We will consider the *Petersen Graph*, which throughout the section will be denoted by X.

For easy reference we number the vertices:



The adjacency matrix of the Petersen graph is then

$$A = \begin{pmatrix} 0&0&1&1&0&0&1&0&0&0 \\ 0&0&0&1&1&0&0&1&0&0 \\ 1&0&0&0&1&0&0&0&1&0 \\ 1&1&0&0&0&0&0&0&0&1 \\ 0&1&1&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&1 \\ 1&0&0&0&0&1&0&1&0&0 \\ 0&1&0&0&0&0&1&0&1&0 \\ 0&0&1&0&0&0&0&1&0&1 \\ 0&0&0&1&0&1&0&0&1&0 \end{pmatrix},$$

where $A_{i,j}$ is the number of edges connecting vertices i and j.

The Petersen graph is 3-regular; so let $q = 3 - 1 = 2$. The Euler characteristic of X is $\chi = 10$. By Theorem 3.1,

$$Z_X(T) = \frac{1}{(1-T^2)^{\chi} \cdot \det(I - AT + qT^2I)}.$$

Which after calculating the determinant becomes

$$Z_X(T) = \frac{1}{(T-1)^6(T+1)^5(2T-1)(2T^2-T+1)^5(2T^2+2T+1)^4}.$$

Using this and Proposition 3.8 we get:

$\sum_{m=1}^{\infty} N_m T^m = TZ'_X(T)/Z_X(T)$

$= 120\,T^5 + 120\,T^6 + 240\,T^8 + 360\,T^9 + 1320\,T^{10} + 2640\,T^{11} + 3360\,T^{12} + 7800\,T^{13} + \cdots$

We can read off the number of closed geodesics of each length. In particular this shows that X has no closed geodesic of length 7 (the loop (1,7,6,10,9,8,7,1) has length 7 but has backtracking).

6.2 Riemann hypothesis

Recall: $\zeta_X(s) = Z_X(q^{-s})$.

So given a pole α of $\zeta_X(s)$, it gives other poles $\alpha + 2\pi n/\log(q)i$. We will give poles of ζ_X such that all of its poles can be obtained this way.

The factor $T - 1 = 0$ gives us a pole at 0.
The factor $T + 1 = 0$ gives us a pole at $0 \pm 4.53..i$.
The factor $T = 1/2$ gives us a pole at 1.
The factor $2T^2 - T + 1$ gives us a pole at $1/2 \pm 1.74...i$.
The factor $2T^2 + 2T + 1$ gives us a pole at $1/2 \pm 3.399..i$.
So it indeed satisfies the Riemann hypotheses!

Of course it should be Ramanujan also.
The matrix A has eigenvalues: $-2, -2, -2, -2, 1, 1, 1, 1, 1, 3$.
Note $2\sqrt{q} = 2\sqrt{2} \approx 2.83$.

6.3 Prime geodesic theorem

By Theorem 4.7, we know that

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < y\} \sim \frac{2^y}{y},$$

as $\mathbb{N} \ni y \to \infty$.

Some calculations (see appendix to find out how these were computed):

y	$\#\{\mathfrak{p} \in \mathcal{P} \mid \ell(\mathfrak{p}) < y\}$	$2^y/y$	$\#\{\mathfrak{p} \in \mathcal{P} \mid \ell(\mathfrak{p}) < y\}/(2^y/y)$
20	58454	52428.8	1.11492
100	12937992133755232438086755376	$1.26765... \times 10^{28}$	1.02063
4000	$\underbrace{3297 \cdots 7100}_{\text{1201 digits}}$	$3.295... \times 10^{1200}$	1.00050

7 Selberg zeta function

This section is based on [1], where full proofs and more can be found.

Let M be a compact Riemann surface with constant negative curvature. [5] We can write M as $\Gamma \backslash \mathbb{H}$, where $\mathbb{H}$ is the Poincaré upper half-plane and Γ is a co-compact subgroup of $PSL_2(\mathbb{R})$.

Definition 7.1. We say that two closed geodesics

$$\gamma, \delta : S^1 = \mathbb{R}/\mathbb{Z} \to M$$

(parametrized with unit speed) are *equivalent* if and only if there exists a constant c such that

$$\gamma(t) = \delta(t + c),$$

for all $t \in S^1$.
Let $\mathscr{C}(M)$ be the set consisting of equivalence classes of closed geodesics on M. For $\gamma \in \mathscr{C}(M)$, we denote the length by $\ell(\gamma)$.

Definition 7.2. An element $\gamma \in \mathscr{C}(M)$ is *prime* if there does not exists a $\gamma_0 \in \mathscr{C}(M)$ and $r \geq 2$ such that

$$\gamma = \gamma_0^r.$$

Let $\mathscr{P}(M) \subset \mathscr{C}(M)$ be the set of primes.

We define a *norm* of geodesics:

$$\begin{array}{rcl} \mathcal{N} : \mathscr{C}(M) & \to & \mathbb{R}^+ \\ \gamma & \mapsto & e^{\ell(\gamma)} \end{array}$$

Motivated by the earlier cases, we define

$$\pi_M(x) = \#\{\mathfrak{p} \in \mathscr{P}(M) : \mathcal{N}(\mathfrak{p}) < x\}.$$

As one would hope, the following is true:

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[5] So the genus should be ≥ 2 by Gauss-Bonnet.

Theorem 7.3.
$$\pi_M(x) \sim \frac{x}{\log x}$$
as $x \to \infty$. $\square$

Corollary 7.4. [6]
$$\#\{\mathfrak{p} \in \mathscr{P}(M) : \ell(\mathfrak{p}) < y\} \sim \frac{e^y}{y}$$
as $y \to \infty$. $\square$

Again by analogy we define:

$$\zeta_M(s) = \prod_{\mathfrak{p} \in \mathscr{P}(M)} \frac{1}{1 - \mathcal{N}(\mathfrak{p})^{-s}}.$$

The function $\zeta_M(s)$ extends to a meromorphic function on $\mathbb{C}$. It satisfies an analogue of the Riemann hypothesis: all but finitely many zeros and poles in the critical strip $0 < \Re s < 1$ lie on the critical line $\Re s = 1/2$. [7]

8 Appendix

8.1 Formula for the number of primes geodesics

We first need a way to effectively compute

$$\xi(y) := \pi_X(q^y) = \#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) < y\}.$$

Proposition 8.1. *For any* $n > 0$,

$$\#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) = n\} = \frac{1}{n} \sum_{d|n} \mu(n/d) N_d.$$

---

[6] It is interesting to note that the asymptotics are independent of the choice of M.

[7] A better zeta function to study is the *Selberg zeta function*

$$Z(s) = \prod_{\mathfrak{p} \in \mathscr{P}(M)} \prod_{v=0}^{\infty} (1 - \mathcal{N}(\mathfrak{p})^{-s-v}),$$

which is entire. Note that $\zeta_M(s) = Z(s+1)/Z(s)$. It is probably preferable to define $Z(s)$ in term of the group Γ.

Proof. Let $P_n = \#\{\mathfrak{p} \in \mathcal{P} | \ell(\mathfrak{p}) = n\}$.

It is easy to see combinatorially that

$$N_n = \sum_{d|n} dP_d.$$

The Proposition follows by the Möbius inversion formula. $\square$

Corollary 8.2.

$$\begin{aligned} \xi(y) &= \sum_{n<y} \frac{1}{n} \sum_{d|n} \mu(n/d) N_d \\ &= \sum_{d<y} \left(\sum_{m<y/d} \frac{\mu(m)}{m} \right) \frac{N_d}{d} \end{aligned}$$

Remark 7. *I find the last formula useful for computations of $\xi(y)$. The N_d can be calculated via the zeta function Z_X (which we know is rational & have a nice formula for!). The values $\sum_{m \leq n} \frac{\mu(m)}{m}$ can be calculated recursively and saved in memory ahead of time (this is especially useful if you are doing computations with different graphs).*

8.2 Heuristics

Remark 8. *In this section we give a heuristic for the key part of our proof of the prime geodesic theorem. In particular, the theorem has a natural heuristic combinatorial explanation. This section can be freely skipped since we have already proven these heuristics.*

Consider some finite connected k-regular graph X ($k > 2$), and set $q = k - 1$ as before.

Heuristic 8.3.

$$\#\{\mathfrak{p} \in \mathcal{P} : \ell(\mathfrak{p}) = l\} \sim g(X) \cdot q^l / l,$$

as $l \to \infty$. (where we should limit to l such that $g(X)|l$).

Proof. Choose one of the $\#V(X)$ points of X. From that vertex choose a direction to start a path (we have k choices here). Now continue moving without backtracking until we have a path of length l (we have $k-1$ possible moves at each vertice). Now we have some random proper walk of length l starting from our initial point. For large l, we have a $\frac{1}{\#V(X)/g(X)}$ probability of our path being closed. We then have a $\frac{k-1}{k}$ probability that the initial and final edges of our walk are different.

So we expect

$$\text{number of proper loops of length } l \sim \frac{\#V(X)\cdot k(k-1)^{l-1}}{\#V(X)/g(X)}\cdot\frac{k-1}{k} = g(X)\cdot q^l.$$

Now we make the heuristic assumption that for large l most of these loops will be prime (this makes sense if you think about it). So now considering equivalence classes of primes we get:

$$\#\{\mathfrak{p}\in\mathcal{P} : \ell(\mathfrak{p}) = l\} \sim g(X)\cdot q^l/l.$$

□

9 Comments

What has been presented is far from the complete story. It is possible to generalize our zeta function to get analogues of L-functions. Dealing with finite graph covers and representations, we can define an analogue of Artin L-functions. Our methods can then be generalized to prove an analogue of the Chebotarev density theorem.

Though we compared the theory for graphs to that of ordinary arithmetic; it is in fact behaves closer to the corresponding theory of curves over finite fields.

The harshest assumption we made was that our graph was regular. If X is not regular, then the definition $Z_X(T) := \prod_{\mathfrak{p}\in\mathcal{P}}\frac{1}{1-T^{\ell(\mathfrak{p})}}$ still makes sense. It can be shown that $Z_X(T)$ is still rational.

For X non-regular we have:
Theorem 9.1. [8] *There is a $\lambda > 0$ such that*

$$\#\{\mathfrak{p}\in\mathcal{P} : \ell(\mathfrak{p}) < g(X)y\} \sim \frac{1}{\lambda^{g(X)}-1}\frac{\lambda^{g(X)\cdot y}}{y}$$

as $\mathbb{N}\ni y\to\infty$. □

The number λ is the Perron-Frobenius eigenvalue of a related operator.

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[8] See [2] for all the details.

These notes were originally written for a lecture, so I did not keep track of my references. However, the following were especially helpful.

References

[1] Buser, Peter, "Geometry and Spectra of Compact Riemann Surfaces"

[2] Hashimoto, Ki-Ichiro, "Artin type L-functions and the density theorem for prime cycles on finite graphs," Internat. J. Math. **3**, (1992), 809-826

[3] Murty, M. Ram, "Ramanujan Graphs," J. Ramanujan Math. Soc. **18**, No.1 (2003), 1-20