# Potential Scattering on the Real Line

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## 1.1 Free Hamiltonian

We start with a systematic analysis of the simplest free scattering problem on $\mathbb{R}$, namely, when the quantum Hamiltonian is given by

$$H_0 = D_x^2 = -\partial_x^2$$

where $D_x = \frac{1}{i}\partial_x$. Its eigenequation is

$$(H_0 - \lambda^2)u = 0 \tag{1.1}$$

with general solution given by

$$u = A\, e^{i\lambda x} + B\, e^{-i\lambda x} \tag{1.2}$$

where $A$ and $B$ are arbitrary constants. Also, the Schrödinger equation

$$\begin{cases} (i\partial_t - H)v = 0 \\ \qquad v|_{t=0} = v_0 \end{cases} \tag{1.3}$$

is solved by

$$v(t,x) = \frac{1}{(4\pi t i)^{\frac{1}{2}}} \int e^{-\frac{(x-y)^2}{4ti}}\, v_0(y)dy\ . \tag{1.4}$$

In particular, when $v_0 = u$ where $u$ as in (1.2), we have

$$v(t,x) = A\, e^{i\lambda x - i\lambda x} + B\, e^{-i\lambda x - i\lambda^2 x}\ . \tag{1.5}$$

Take $\lambda > 0$ and write the phase as

$$\lambda(x - \lambda t) \ \text{ and } \ \lambda(x + \lambda t)\ .$$

We can think of the first term in (1.5) as a wave moving to the right and the second term as a wave moving to the left. Consequently, we call

$$Ae^{i\lambda x} \text{ for } x < 0\ , \qquad Be^{-i\lambda x} \text{ for } x > 0$$

the incoming terms and

$$Ae^{i\lambda x} \text{ for } x > 0\ , \qquad Be^{-i\lambda x} \text{ for } x < 0$$

the outgoing terms.

$$\begin{array}{ccc} e^{i\lambda x} & \longrightarrow \quad \longrightarrow & \\ & - - - - - - & x \\ e^{-i\lambda x} & \longleftarrow \quad \longleftarrow & \end{array}$$

Figure 1

Hence, the solution $e_+(x,\lambda) := e^{i\lambda x}$ of (1.1) is incoming for $x < 0$ and outgoing for $x > 0$. Similarly, the solution $e_-(x,\lambda) := e^{-i\lambda x}$ of (1.1) has the opposite property. We call them plane waves. Clearly, there exists no solution of (1.1) with only incoming terms or only outgoing terms for $\lambda \neq 0$. However, for $\lambda = 0$, the solution $u \equiv 1$ is considered both incoming and outgoing. Although the physical intuition underlying our incoming and outgoing convention makes sense only for $\lambda > 0$, we will take the same convention for all $\lambda \in \mathbb{C}$. Note that this amounts to a convention of taking the square root of the energy $\lambda^2$.

Next we consider the equation

$$(H_0 - \lambda^2)u = f \quad \text{for} \quad f \in C_0^\infty(\mathbb{R}). \tag{1.6}$$

It has a unique outgoing and a unique incoming solution given by

$$u_\pm(x,\lambda) = \pm\frac{i}{2\lambda}\int f(y)\, e^{\pm i\lambda|x-y|}\, dy \tag{1.7}$$

where the plus sign gives the outgoing solution and the minus sign gives the incoming one. Note that the uniqueness follows from the fact that there is no incoming or outgoing solution to the eigenequation (1.1) as mentioned above.

An alternative characterization of the outgoing or incoming solution $u_\pm$ of (1.6) is that

$$u_+(x,\lambda) \in L^2(\mathbb{R}) \quad \text{for } \mathrm{Im}\lambda > 0$$

and

$$u_-(x,\lambda) \in L^2(\mathbb{R}) \quad \text{for } \mathrm{Im}\lambda < 0.$$

Also, we have

$$u_+(x,\lambda) = \overline{u_-(x,\lambda)} = u_-(x,-\lambda). \tag{1.8}$$

We can now define the outgoing resolvent of $H_0$, $R_0(\lambda)$, by

$$u_+ = R_0(\lambda) f \quad \text{for} \quad f \in C_0^\infty(\mathbb{R}) .$$

and $R_0(-\lambda)$ is then defined to be the incoming resolvent which satisfies

$$u_- = R_0(-\lambda) f \quad \text{for} \quad f \in C_0^\infty(\mathbb{R}) \ .$$

by (1.8). Note that (1.7) implies

$$R_0(\lambda)(x,y) = \frac{i}{2\lambda}\, e^{i\lambda|x-y|} \ . \tag{1.9}$$

Clearly, the outgoing resolvent $R_0(\lambda)$ is bounded on $L^2(\mathbb{R})$ for $\mathrm{Im}\ \lambda > 0$. Moreover, its norm is given by

$$\|R_0(\lambda)\|_{L^2\to L^2} = \frac{1}{d(\mathbb{R}_+,\, \lambda^2)} \ . \tag{1.10}$$

where $d(\mathbb{R}_+,\, \lambda^2)$ denotes the distance between $\lambda^2$ and the positive real axis $\mathbb{R}_+$. This follows from standard facts in the spectral theory of self-adjoint operator. In our case, it can be seen directly from the Plancherel Theorem. Indeed, for $f \in C_0^\infty(\mathbb{R})$, we have

$$R_0(\lambda) f = \mathcal{F}^{-1}\left( \frac{1}{(\ \cdot\ )^2 - \lambda^2}\, \mathcal{F} f \right)$$

where $\mathcal{F}$ is the Fourier transform. Since

$$\|M_{\frac{1}{(\ \cdot\ )^2-\lambda^2}}\|_{L^2\to L^2} = \sup_{\xi\in\mathbb{R}}\ \frac{1}{\xi^2 - \lambda^2} = \frac{1}{d(\mathbb{R}_+,\, \lambda^2)}$$

with $M_g(u) = gu$ denoting the multiplication operator, (1.10) follows.

Also, by (1.9), $R_0(\lambda)$ as an operator

$$R_0(\lambda) : L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R})$$

has a meromorphic extension to $\lambda \in \mathbb{C}$ with a simple pole at $\lambda = 0$. In fact, we have

$$R_0(\lambda) = \frac{P}{\lambda} + Q(\lambda)$$

with $(Pf)(x) = \dfrac{i}{2}\displaystyle\int_{\mathbb{R}} f(y)dy$ for $f \in L^2_{\mathrm{comp}}(\mathbb{R})$ and $Q(\lambda) : L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R})$ being entire in $\lambda \in \mathbb{C}$.

Formally, $P = \phi \otimes \bar{\phi}$ where $\phi(x) = \dfrac{e^{\pi i/4}}{\sqrt{2}}$ can be regarded as an outgoing solution to (1.1) at $\lambda = 0$.

The spectral decomposition of $H_0$ can be given by the Fourier transform as follows (here we use the same symbol of the operator to denote its Schwartz kernel),

$$H_0(x,y) = D_x^2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda^2\, e^{i\lambda(x-y)}\, dy := \int_0^{\infty} \lambda^2 dE_\lambda(x,y)\ . \tag{1.11}$$

Thus the spectral measure $dE_\lambda$ is given by

$$\begin{aligned} dE_\lambda(x,y) &= \frac{1}{2\pi}(e^{i\lambda(x-y)} + e^{-i\lambda(x-y)})d\lambda \\ &= \frac{1}{2\pi}(e_+(x,\lambda)\overline{e_+(y,\lambda)} + e_-(x,\lambda)\overline{e_-(y,\lambda)})d\lambda \\ &= \frac{\lambda}{i\pi}(R_0(\lambda) - R_0(-\lambda))(x,y)d\lambda \end{aligned} \tag{1.12}$$

where the last equality follows from (1.9). Note that (1.12) is a special case of a general result in functional analysis, namely the Stone's Formula, and of the spectral decomposition in terms of generalized eigenfunctions. To see this, we have to clarify the convention. Write

$$H_0 = \int_0^{\infty} z dE_z = \int_0^{\infty} \lambda^2 dE_\lambda \qquad (z = \lambda^2)$$

as usual (note that $H_0$ is a nonnegative operator). Thus $dE_z = dE_\lambda$. In our incoming and outgoing convention, for $\lambda > 0$, $R_0(\lambda) = R_0(z + i0)$ and $R_0(-\lambda) = R_0(z - i0)$. Hence the usual Stone's Formula,

$$dE_z = \frac{1}{2\pi i}(R_0(z+i0) - R_0(z-i0))dz$$

coincides with our formula (1.12).

Next we mention the connection with the wave equation:

$$(D_t^2 - D_x^2)U = f\ . \tag{1.13}$$

It can be solved by using the advanced and retarded fundamental solutions $E_\pm$ which are characterized uniquely by the following support property.

$$\begin{aligned} (D_t^2 - D_x^2)E_\pm(t,x,y) &= \delta_0(t)\delta_y(x) \\ E_\pm(t,x,y) &= 0 \ \text{ for } \ \pm t < 0 \end{aligned} \tag{1.14}$$

Indeed, we have

$$E_\pm(t,x,y) = \begin{cases} \frac{1}{2} & \pm t > |x-y| \\ 0 & \text{otherwise.} \end{cases}$$

Their relation with the outgoing and incoming resolvents $R_0(\pm\lambda)$ is then given by

$$E_{\pm}(t,x,y) = \frac{1}{2\pi}\int R_0(\pm\lambda)(x,y)e^{-i\lambda t}d\lambda \ . \tag{1.15}$$

Finally, we remark that our incoming and outgoing convention is motivated by the Schrödinger equation and is different from the wave equation approach.

## 1.2 Perturbed Hamiltonian and Distorted Plane Waves

We now consider the perturbed Hamiltonian on $\mathbb{R}$:

$$H_V = D_x^2 + V \quad \text{for} \quad V \in L^\infty_{\text{comp}}(\mathbb{R})$$

We would like to establish the same results we had for $H_0$ in section 1.1.

We will first show the existence of the resolvent $R_V(\lambda)$ which satisfies, for $f \in C_0^\infty(\mathbb{R})$,

$$(H_V - \lambda^2)R_V(\lambda)f = f \tag{1.16}$$

with $R_V(\lambda)f$ being outgoing. Here, the meaning of outgoing (or incoming) can be taken as that of section 1.1 since $R_V(\lambda)f$ solves (1.1) for large $x$.

**Theorem 1.1** *The operator*

$$R_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R})$$

*satisfying* (1.16) *exists as a meromorphic function of operators for* $\lambda \in \mathbb{C}$ *and it has no pole for* $\lambda \in \mathbb{R}\setminus\{0\}$.

**Proof.** For $\text{Im}\ \lambda > 0$, by applying the operator $H_V - \lambda^2$ to the free resolvent, we get

$$(D_x^2 + V - \lambda^2)R_0(\lambda) = I + VR_0(\lambda) \ . \tag{1.17}$$

By (1.10), we have

$$\|VR_0(\lambda)\|_{L^2\to L^2} << 1 \qquad \text{for} \quad \text{Im}\ \lambda >> 0 \ .$$

Thus, for $\text{Im}\ \lambda >> 0$,

$$(I + VR_0(\lambda))^{-1} = \sum_{k=0}^{\infty}(-VR_0(\lambda))^k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

exists and is holomorphic in $\lambda$.

For $\text{Im}\ \lambda >> 0$, let

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1} \ . \tag{1.18}$$

Clearly, $R_V(\lambda)$ is a holomorphic family of bounded operators on $L^2(\mathbb{R})$ which satisfies (1.16).

We need to show that as operators $R_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R})$, $R_V(\lambda)$ continues meromorphically from Im $\lambda >> 0$ to the whole complex plane. This is the same as extending

$$R_V(\lambda)\rho : L^2(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R})$$

meromorphically to $\mathbb{C}$ for any $\rho \in C_0^\infty(\mathbb{R})$ where $\rho$ denotes the corresponding multiplication operator. Note that once the meromorphic continuation is established, it does not depend on $\rho$ in the following sense:

$$\begin{aligned} &\text{if} \quad \rho_1 \in C_0^\infty(\mathbb{R}) \quad \text{with } \rho_1 \equiv 1 \ \text{ on } \ \text{Supp}\ \rho, \\ &\text{then we have } \ (R_V(\lambda)\rho_1)\rho = R_V(\lambda)\rho \ . \end{aligned} \tag{1.19}$$

In fact, this is obviously true for Im $\lambda >> 0$, and then for $\lambda \in \mathbb{C}$ by meromorphic continuation.

Next we observe that, for Im $\lambda >> 0$ and $\rho \in C_0^\infty(\mathbb{R})$ with $\rho V \equiv V$,

$$R_V(\lambda)\rho = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1} \tag{1.20}$$

by (1.18) and the equality

$$(I + VR_0(\lambda))^{-1}\rho = \rho(I + VR_0(\lambda)\rho)^{-1} \ .$$

Hence, the meromorphic continuation of $R_V(\lambda)$ is reduced to that of $(I + VR_0(\lambda)\rho)^{-1}$. This in turn follows from the Analytic Fredholm Theory in the Appendix, once we note that $VR_0(\lambda)\rho = V\rho R_0(\lambda)\rho$ is a meromorphic family of compact operators on $L^2(\mathbb{R})$ as $\rho R_0(\lambda)\rho : L^2(\mathbb{R}) \to H^2_{\text{comp}}(\mathbb{R})$ for $\lambda \in \mathbb{C}\backslash\{0\}$ is compact and $V : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ bounded.

To complete the proof of Theorem 1.1 it remains to show that there is no pole for $R_V(\lambda)$ on $\mathbb{R}\backslash\{0\}$. This will come in several steps. We start with

**Proposition 1.1** *If $R_V(\lambda)$ has a pole at $\lambda = \lambda_0 \neq 0$ and write*

$$R_V(\lambda) = \frac{P_N}{(\lambda - \lambda_0)^N} + \frac{P_{N-1}}{(\lambda - \lambda_0)^{N-1}} + \cdots + \frac{P_1}{\lambda - \lambda_0} + Q(\lambda) \tag{1.21}$$

*for $\lambda$ near $\lambda_0$ where $Q(\lambda)$ is holomorphic at $\lambda_0$, then $u \in P_N(L^2_{\text{comp}}(\mathbb{R}))$ is an outgoing solution of $(H_V - \lambda_0^2)u = 0$.*

**Proof.** First, note that the expansion (1.21) follows from meromorphy of $R_V(\lambda)$. Next, by applying the operator $(\lambda-\lambda_0)^N(H_V-\lambda^2)$ to $R_V(\lambda)$ and then putting $\lambda=\lambda_0$, we see that

$$(H_V-\lambda_0^2)P_N\equiv 0\ .$$

It remains to show that elements in $P_N(L^2_{\text{comp}}(\mathbb{R}))$ are always outgoing. For this, we let $\rho\in C_0^\infty(\mathbb{R})$ with $\rho V\equiv V$. By (1.20), we can write

$$(I+VR_0(\lambda)\rho))^{-1}=\frac{\tilde{P}_N}{(\lambda-\lambda_0)^N}+\cdots+\frac{\tilde{P}_1}{\lambda-\lambda_0}+\tilde{Q}(\lambda)$$

for $\lambda$ near $\lambda_0$ where $\tilde{P}_j,\tilde{Q}(\lambda):L^2(\mathbb{R})\to L^2(\mathbb{R})$, $j=1,\dots N$, and $\tilde{Q}(\lambda)$ is holomorphic at $\lambda_0$. Then (1.20) also implies that

$$P_N(\rho L^2(\mathbb{R}))=R_0(\lambda)\rho\tilde{P}_N(L^2(\mathbb{R}))\subset R_0(\lambda)(L^2_{\text{comp}}(\mathbb{R}))$$

which means that elements of $P_N(L^2_{\text{comp}}(\mathbb{R}))$ are outgoing.

Next, we prove that there is no outgoing solution to $(H_V-\lambda^2)u=0$ for $\lambda\in\mathbb{R}\backslash\{0\}$. This follows from the following proposition.

**Proposition 1.2** *Suppose* $(H_V-\lambda^2)u=0$ *for* $\lambda\in\mathbb{R}\backslash\{0\}$ *and let*

$$u(x)=\begin{cases}A_+\,e^{i\lambda x}+B_-\,e^{-i\lambda x} & \text{for } x>>0\\ A_-\,e^{i\lambda x}+B_+\,e^{-i\lambda x} & \text{for } x<<0\end{cases}$$

*Then*

$$|A_+|^2+|B_+|^2=|A_-|^2+|B_-|^2 \tag{1.22}$$

**Proof.** Since $\lambda$ is real, $\bar{u}$ also satisfies the equation. Thus, the Wronskian of $u$ and $\bar{u}$

$$W(u,\bar{u})=\begin{vmatrix} u & u' \\ \bar{u} & \bar{u}'\end{vmatrix}=\begin{cases}-2i\lambda(|A_+|^2-|B_-|^2) & \text{for } x>>0\\ -2i\lambda(|A_-|^2-|B_+|^2) & \text{for } x<<0\end{cases}$$

is constant. (1.22) follows immediately.

**Proposition 1.3** *Suppose* $u\in L^\infty(\mathbb{R})$ *satisfies* $(D_x^2+W)u=0$ *for* $W\in L^\infty(\mathbb{R})$ *and* supp $u\subset[0,\infty)$. *Then* $u\equiv 0$.

**Proof.** Fix $h > 0$. Let $v = e^{-\frac{x}{h}}u$. Then the boundedness and the support property of $u$ imply that $v \in L^2(\mathbb{R})$. Now we have

$$\begin{aligned}
\|e^{-\frac{x}{h}}(hD_x)^2 e^{\frac{x}{h}} v\|_{L^2} &= \|(h^2D_x^2 - 2ihD_x - 1)v\|_{L^2} \\
&= \|(h\xi - i)^2 \hat{v}\|_{L^2} \qquad \text{by Plancherel formula} \\
&\geq \|\hat{v}\|_{L^2} \qquad \text{since } |h\xi - i| \geq 1 \ \text{ as } \ h\xi \in \mathbb{R} \\
&= \|v\|_{L^2} \ .
\end{aligned}$$

Changing back to $u$, we get

$$\begin{aligned}
\|e^{-\frac{x}{h}}u\|_{L^2} &\leq \|e^{-\frac{x}{h}}h^2D_x^2u\|_{L^2} \\
&= \|e^{-\frac{x}{h}}h^2Wu\|_{L^2} \\
&\leq \|W\|_{L^\infty} h^2 \, \|e^{-\frac{x}{h}}u\|_{L^2} \ .
\end{aligned}$$

Taking $h^2 < \|W\|_{L^\infty}^{-1}$, we obtain $e^{-\frac{x}{h}}u \equiv 0$, i.e. $u \equiv 0$.

Now we can finish the proof of Theorem 1.1. In fact, assume that there is a pole of $R_V(\lambda)$ at $\lambda_0 \in \mathbb{R}\backslash\{0\}$. Proposition 1.1 implies that there is a nonzero outgoing solution $u$ of the equation $(H_V - \lambda_0^2)u = 0$. Proposition 1.2 then implies that $u$ is vanishing outside some compact set in $\mathbb{R}$. But Proposition 1.4 says that $u \equiv 0$ which is a contradiction.

**Remark.** Proposition 1.2 is the only part of the proof of Theorem 1.1 which is simpler in dimension 1, all the remaining parts work in higher dimensions.

With the construction of the resolvent $R_V(\lambda)$ in place, we can now define and obtain the distorted plane waves which constitute the continuous spectrum of $H_V$.

**Proposition 1.4** *For $\lambda \in \mathbb{R}\backslash\{0\}$, there exist unique solutions $e_\pm(x, \lambda)$ to*

$$(H_V - \lambda_0^2)u = 0 \tag{1.23}$$

*satisfying $e_\pm(x, \lambda) = e^{\pm i\lambda x} +$ outgoing terms.*

**Proof.** For $\lambda \in \mathbb{R}\backslash\{0\}$, put

$$e_\pm(x, \lambda) = e^{\pm i\lambda x} - R_V(\lambda)(V\, e^{\pm i\lambda x}) \tag{1.24}$$

which makes sense because of Theorem 1.1. Clearly, $e_\pm(x, \lambda)$ satisfies the equation (1.23) and the last term in (1.24) is outgoing thanks to (1.20). Uniqueness again follows from nonexistence of outgoing solution to (1.23) proved in Theorem 1.1.

To establish the analogue of (1.12), we use a standard ODE technique to represent $R_V(\lambda)$ in terms of $e_\pm(x,\lambda)$. First, recall that, in general the fundamental solution of the ODE

$$(a\partial_x^2 + b(x))E(x,y) = \delta_y(x)$$

can be written in terms of any two linearly independent solutions $\psi_j$, $j = 1,2$ of

$$(a\partial_x^2 + b(x))\psi_j = 0 \qquad j = 1,2 \ .$$

Namely,

$$E(x,y) = \frac{1}{aW}\left(\psi_1(x)\psi_2(y)(x-y)_+^0 + \psi_1(y)\psi_2(x)(x-y)_-^0\right) \tag{1.25}$$

where $W = \begin{vmatrix} \psi_1 & \psi_1' \\ \psi_2 & \psi_2' \end{vmatrix}$ is the Wronskian of $\psi_1, \psi_2$ and

$$(x-y)_\pm^0 = \begin{cases} 1 & \pm x > \pm y \\ 0 & \text{otherwise} \end{cases}$$

denote the Heaviside functions.

We now apply (1.25) to the operator $H_V - \lambda^2$ for $\lambda \in \mathbb{R}\backslash\{0\}$ and solutions $\psi_1 = e_+$, $\psi_2 = e_-$. Note that, for fixed $y$, both $e_+(x,\lambda)(x-y)_+^0$ and $e_-(x,\lambda)(x-y)_-^0$ are outgoing and hence (1.25) gives us the outgoing resolvent.

To write down $R_V(\lambda)(x,y)$ explicitly, we need to compute the Wronskian of $e_+$ and $e_-$. First, observe that, by the characterizing properties of $e_\pm$, we can write

$$e_\pm(x,\lambda) = \begin{cases} T_\pm(\lambda)e^{\pm i\lambda x} & \text{for} \quad \pm x >> 0 \\ e^{\pm i\lambda x} + R_\pm(\lambda)e^{\mp i\lambda x} & \text{for} \quad \pm x << 0 \end{cases} \tag{1.26}$$

Computing the Wronskian $W$ of $e_+$ and $e_-$ for large $x$ and large negative $x$ respectively, we get

$$W = -2i\lambda T_+(\lambda) = -2i\lambda T_-(\lambda) \ .$$

In particular,

$$T_+(\lambda) = T_-(\lambda) := T(\lambda) \ . \tag{1.27}$$

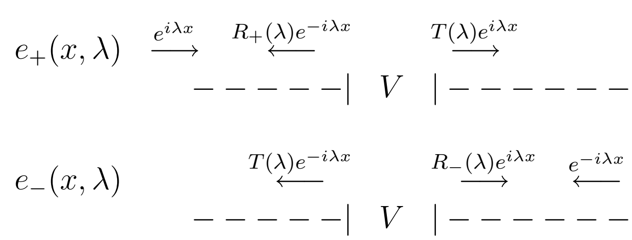

Figure 2

$T(\lambda)$ is called the transmission coefficient and $R_{\pm}(\lambda)$ the reflection coefficients. Now, we can write down the expression for $R_V(\lambda)$ in terms of $e_{\pm}$ from (1.25). For $\lambda \in \mathbb{R}\backslash\{0\}$, we have

$$R_V(\lambda)(x,y) = \frac{1}{2i\lambda T(\lambda)}\left(e_+(x,\lambda)e_-(y,\lambda)(x-y)_+^0 + e_+(y,\lambda)e_-(x,\lambda)(x-y)_-^0\right) . \tag{1.28}$$

This implies the following useful asymptotics

$$R_V(\lambda)(\pm r, y) = \frac{1}{2i\lambda}\, e^{\pm i\lambda r}\, e_{\mp}(y,\lambda) \qquad \text{for} \;\; r >> 0 . \tag{1.29}$$

The spectral decomposition of $H_V$ is now given by

**Theorem 1.2** *Let $e_{\pm}$ be given by Proposition 1.4. Then*

$$\delta(x-y) = \frac{1}{2\pi}\int_0^\infty \left(e_+(x,\lambda)\overline{e_+(y,\lambda)} + e_-(x,\lambda)\overline{e_-(y,\lambda)}\right) d\lambda + \sum_{j=1}^N e_j(x)\overline{e_j(y)}$$

*and*

$$H_V(x,y) = \frac{1}{2\pi}\int_0^\infty \lambda^2\left(e_+(x,\lambda)\overline{e_+(y,\lambda)} + e_-(x,\lambda)\overline{e_-(y,\lambda)}\right) d\lambda + \sum_{j=1}^N E_j e_j(x)\overline{e_j(y)} \tag{1.30}$$

*where $E_j = \lambda_j^2$, $j = 1,\ldots,N$, $\lambda_j' s$ are the poles of $R_V(\lambda)$ for* $\mathrm{Im}\,\lambda > 0$*, and $(H_V - E_j)e_j = 0$ with $\|e_j\|_{L^2} = 1$.*

**Proof.** According to the Appendix, Theorem 1.1 and the boundedness of $V$, $H_V$ acting on $C_0^\infty(\mathbb{R})$ has a self-adjoint extension on $L^2(\mathbb{R})$ whose spectrum consists of finitely many negative eigenvalues (with multiplicity) and a continuous part $[0,\infty)$. Hence

$$H_V = \sum_{j=1}^N E_j e_j \otimes \bar{e}_j + \int_0^\infty z dE_z = \sum_{j=1}^N E_j e_j \otimes \bar{e}_j + \int_0^\infty \lambda^2 dE_\lambda \tag{1.31}$$

where the $E_j$'s are the eigenvalues of $H_V$ and the $e_j$'s are the corresponding normalized eigenfunctions, i.e. $(H_V - E_j)e_j = 0$, $\|e_j\|_{L^2} = 1$. (In the second equality, we use the substitution $z = \lambda^2$.)

In our convention of taking square root of $z$, $E_j = \lambda_j^2$ where the $\lambda_j$'s are the poles of $R_V(\lambda)$ for Im $\lambda > 0$. To compute $dE_\lambda$ we use the Stone's Formula

$$dE_\lambda = \frac{\lambda}{\pi i}(R_V(\lambda) - R_V(-\lambda))d\lambda . \tag{1.32}$$

We want to express the right-hand side by the distorted plane waves $e_\pm$. For this we use

$$(D_x^2 + V - \lambda^2)R_V(\pm\lambda)(x,y) = \delta(x-y)$$

and the symmetry of $R_V(\pm\lambda)(x,y)$ with respect to $x, y$ to write for fixed $x, y$ and large $r$,

$$\begin{aligned}
& (R_V(\lambda) - R_V(-\lambda))(x,y) \\
= & \int_{-r}^{r} [R_V(\lambda)(x,y')(D_{y'}^2 R_V(-\lambda)(y',y)) - (D_{y'}^2 R_V(\lambda)(x,y'))R_V(-\lambda)(y',y)]dy' \\
= & [R_V(\lambda)(x,y')D_{y'}R_V(-\lambda)(y',y) - D_{y'}R_V(\lambda)(x,y')R_V(-\lambda)(y',y)]|_{y'=-r}^{y'=r} \\
= & \frac{i}{2\lambda}(e_+(x,\lambda)\overline{e_+(y,\lambda)} + e_-(x,\lambda)\overline{e_-(y,\lambda)}\,)
\end{aligned} \tag{1.33}$$

by (1.29). Theorem 1.2 now follows by putting (1.33) into (1.31) and (1.32).

As at the end of section 1.1, we can also consider the relation to the wave equation. The advanced and retarded fundamental solutions $E_\pm$ are again characterized by

$$\begin{aligned}
(D_t^2 - (D_x^2 + V))E_\pm(t,x,y) &= \delta_0(t)\delta_y(x) \\
E_\pm(t,x,y) &= 0 \quad \text{for} \quad \pm t < 0 \,.
\end{aligned}$$

Again we have

$$E_\pm(t,x,y) = -\frac{1}{2\pi}\int R_V(\pm\lambda)(x,y)e^{-it\lambda}d\lambda \,.$$

We have more or less established in the perturbed case all the results we proved in section 1.1. We end this section by discussing a class of intertwining operators $A_\pm$ satisfying $H_V A_\pm = A_\pm H_0$ which will be useful later. More precisely, we want to find distributions $A_\pm(x,y)$ satisfying

$$\begin{aligned}
(D_x^2 + V)A_\pm(x,y) &= D_y^2 A_\pm(x,y) \\
A_\pm(x,y) &= \delta(x-y) \qquad \pm x >> 0 \,.
\end{aligned} \tag{1.34}$$

To show their existence, we will first construct solutions of the stationary equation, for $\lambda \in \mathbb{R}$,

$$\begin{aligned}
(D_x^2 + V - \lambda^2)\phi_\pm(x,\lambda) &= 0 \\
\phi_\pm(x,\lambda) &= e^{-i\lambda x} \quad \pm x >> 0 \,.
\end{aligned} \tag{1.35}$$

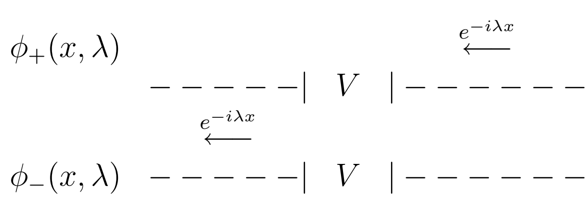

Figure 3

**Lemma 1.1** *There exist unique solutions $\phi_\pm(x,\lambda)$ to (1.35) and for fixed $x$, $\phi_\pm(x,\lambda)$ are tempered functions in $\lambda \in \mathbb{R}$.*

**Proof.** Put

$$\phi_\pm(x,\lambda) = \frac{1}{T(\mp\lambda)}\ e_\pm(x,\mp\lambda) \tag{1.36}$$

where $T(\lambda)$ is defined in (1.26) and (1.27). Then $\phi_\pm(x,\lambda)$ are meromorphic in $\lambda \in \mathbb{C}$.

We claim that $\phi_\pm(x,\lambda)$ are holomorphic on $\mathbb{R}$. Indeed, for $\lambda \in \mathbb{R}\backslash\{0\}$, $T(\lambda) \neq 0$ otherwise $e_\pm(x,\lambda)$ would be identically zero by Proposition 1.3. Next, suppose $\phi_\pm(x,\lambda)$ has a pole of order $m > 0$ at $\lambda = 0$. Then $\lambda^m \phi_\pm(x,\lambda)$ is holomorphic for $\lambda$ near 0. Let

$$\tilde\phi_\pm(x) = \lim_{\lambda\to 0} \lambda^m \phi_\pm(x,\lambda)\ ,$$

then $\tilde\phi_\pm(x)$ is a solution of

$$\begin{aligned} (D_x^2 + V)\tilde\phi_\pm(x) &= 0 \\ \tilde\phi_\pm(x) &= 0 \quad \pm x >> 0\ . \end{aligned}$$

Hence $\tilde\phi_\pm(x) \equiv 0$ by Proposition 1.3 which is a contradiction. Although we don't need this fact in the proof of the Lemma, we remark that the above argument actually shows that $\phi_\pm(x,\lambda)$ is defined and holomorphic for $\lambda \in \mathbb{C}$. As $\phi_\pm(x,\lambda)$ clearly satisfy (1.35), it remains to verify the temperedness of $\phi_\pm(x,\lambda)$ as $|\lambda| \to \infty$. For that, recall by (1.9),

$$|(VR_0(\lambda)\rho)(x,y)| \leq \frac{1}{|\lambda|} \quad \text{for} \quad |\lambda| >> 0$$

where $\rho \in C_0^\infty(\mathbb{R})$ with $\rho V \equiv V$; also from (1.20) we have

$$R_V(\lambda)\rho = R_0(\lambda)\rho \sum_{k=0}^{\infty} (R_0(\lambda)V)^k\ .$$

(1.29) then shows that

$$|e_\pm(x,\lambda)| \leq C \quad \text{for} \quad |\lambda| >> 0\ . \tag{1.37}$$

Moreover, we have

$$|T(\lambda)|^{-1} \leq C \quad \text{for} \quad |\lambda| >> 0\ .$$

This follows from a similar argument as

$$T(\lambda) = 1 - e^{-i\lambda x} R_V(\lambda)(Ve^{i\lambda\cdot}) \quad \text{for} \quad |\lambda| >> 0$$

which implies

$$T(\lambda) = 1 + O\left(\frac{1}{|\lambda|}\right) \quad \text{as} \;\; |\lambda| \to \infty \; . \tag{1.38}$$

This completes the proof of our lemma.

**Proposition 1.5** *There exist unique solutions* $A_\pm(x,y)$ *to* (1.34). *Moreover, they satisfy the following properties:*

(a) Supp $A_\pm(x,y) \subset \{(x,y) \in \mathbb{R}^2 : \mp x \geq \mp y\}$

(b) $\delta_y A_-(x,y) = X(y-x) + Y(x+y) \qquad x >> 0\,.$

*Here* $X, Y$ *are distributions with compact support with*

$$\begin{aligned} &\text{supp } X \subset [-2(b-a), 0] \\ &\text{supp } Y \subset [2a, 2b] \end{aligned} \tag{1.39}$$

*where* $[a,b] = \text{ch supp } V\,.$

**Proof.** Rewriting (1.34) slightly, we have

$$\begin{cases} D_x^2 - (D_y^2 - V(x))A_\pm(x,y) = 0 \\ A_\pm(x,y) = \delta(x-y) & \text{for} \quad \pm\, x >> 0 \end{cases}$$

Thus $A_\pm(x,y)$ satisfies the wave equation with $x$ taking the place of time (this choice is dictated by the forcing condition imposed). The uniqueness part then follows from the energy estimates of the wave equation proved in the Appendix.

For the existence part, we put

$$A_\pm(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_\pm(x,\lambda) e^{i\lambda y} d\lambda \;, \tag{1.40}$$

which is well defined, thanks to Lemma 1.1.

Now $A_\pm(x,y)$ satisfies (1.34) because of (1.35). (a) is then a direct consequence of the energy estimates, that $\partial_y A_-(x,y)$ is of the form given in (b) for $x >> 0$ is simply because it satisfies the wave equation there:

$$(D_x^2 - D_y^2)\partial_y A_-(x,y) \;=\; 0 \quad \text{for}\, x >> 0.$$

The support properties of $X$ and $Y$ can now be seen from Figure 4 which shows the support of $\partial_y A_-(x,y)$ (the region enclosed by the thick lines) and with the supports of $X(y-x)$ and $Y(y+x)$ indicated.

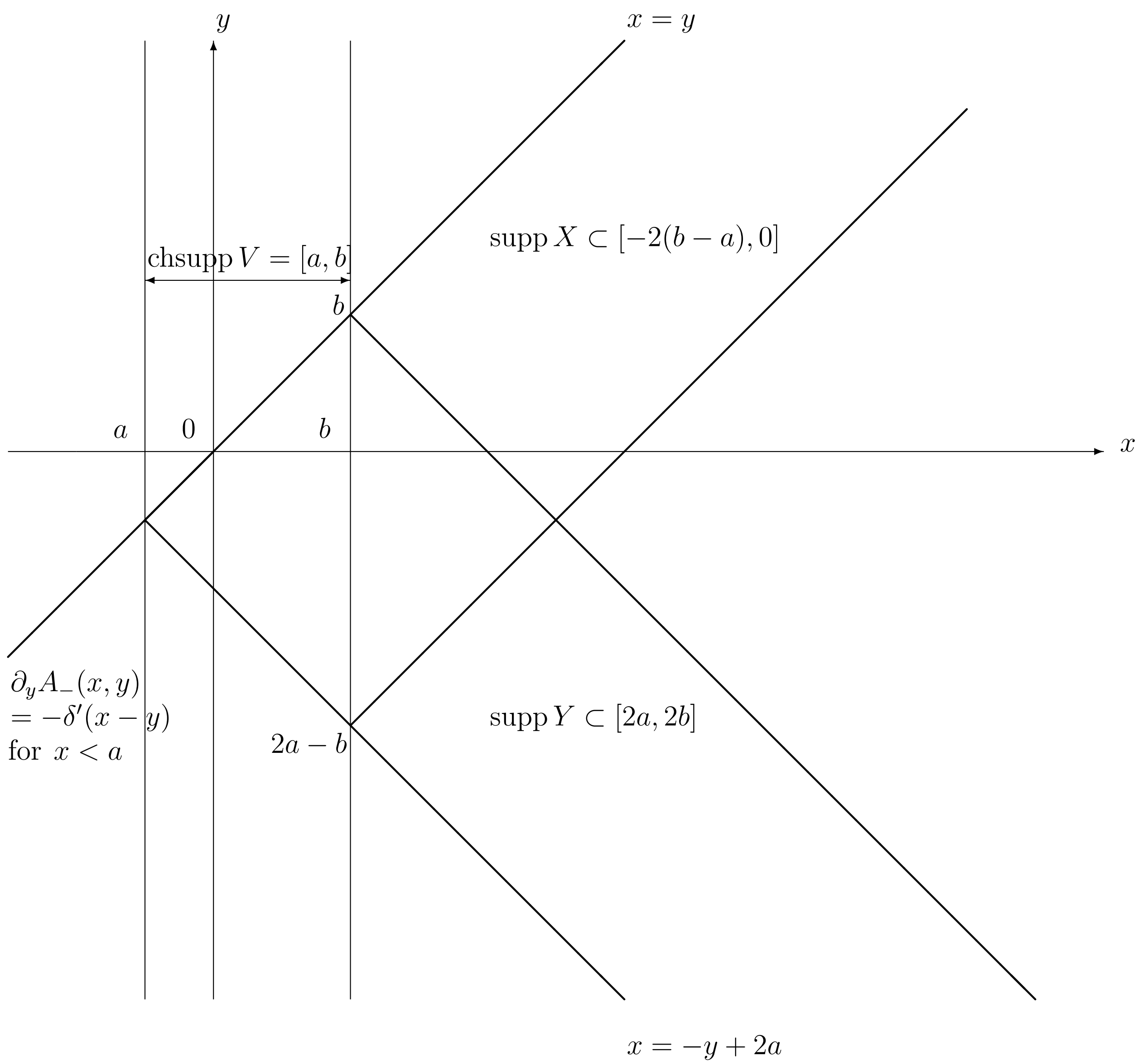


Figure 4

**Remark.** A direct construction of the intertwining kernels $A_{\pm}(x, y)$ is important in inverse problems.

## 1.3 Scattering Matrix and Wave Operators

We start with the definition of the scattering matrix associated to the Hamiltonian

$$H_V = D_x^2 + V \qquad V \in L^\infty_{\text{comp}}(\mathbb{R})$$

in section 1.2. For any solution $u$ of

$$(D_x^2 + V - \lambda^2)u \;=\; 0\;, \tag{1.41}$$

it has the expansion

$$u(x,\lambda) = \begin{cases} A_+\, e^{i\lambda x} + B_-\, e^{-i\lambda x} & \text{for } \; x >> 0 \\ A_-\, e^{i\lambda x} + B_+\, e^{-i\lambda x} & \text{for } \; x << 0\;. \end{cases} \tag{1.42}$$

$$u(x,\lambda) \qquad \overset{B_+\,e^{-i\lambda x}}{\longleftarrow} \quad \overset{A_-\,e^{i\lambda x}}{\longrightarrow} \qquad \overset{B_-\,e^{-i\lambda x}}{\longleftarrow} \quad \overset{A_+\,e^{i\lambda x}}{\longrightarrow}$$

$$- - - - - -| \quad V \quad | - - - - - -$$

Figure 5

Then the scattering matrix is defined to be the operator which maps the incoming coefficients to the outgoing coefficients, i.e. $S(\lambda) : \mathbb{C}^2 \to \mathbb{C}^2$.

$$\begin{pmatrix} A_- \\ B_- \end{pmatrix} \mapsto \begin{pmatrix} A_+ \\ B_+ \end{pmatrix} \tag{1.43}$$

**Theorem 1.3** *The matrix $S(\lambda)$ is meromorphic for $\lambda \in \mathbb{C}$ where poles with* $\operatorname{Im}\,\lambda > 0$ *correspond to the square roots of the eigenvalues of $H_V$. In the notation of Proposition 1.5 we have*

$$S(\lambda) = \begin{pmatrix} \frac{i\lambda}{\hat{X}(\lambda)} & \frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} \\ & \\ \frac{\hat{Y}(-\lambda)}{\hat{X}(\lambda)} & \frac{i\lambda}{\hat{X}(\lambda)} \end{pmatrix} \tag{1.44}$$

*where $\hat{X}$ denotes the Fourier transform of $X$ and*

$$S(\lambda)S(\bar{\lambda})^* = S(\lambda)JS(-\lambda)J = I \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.45}$$

**Proof.** To find $S(\lambda)$, we use the two linearly independent solutions $\phi_\pm(x,\lambda)$ of (1.41) given by Lemma 1.1. Write

$$\phi_-(x,\lambda) = \begin{cases} A(\lambda)e^{i\lambda x} + B(\lambda)e^{-i\lambda x} & \text{for } \; x >> 0 \\ e^{-i\lambda x} & \text{for } \; x << 0 \end{cases}$$

and

$$\phi_+(x,\lambda) = \begin{cases} e^{-i\lambda x} & \text{for } \ x >> 0 \\ C(\lambda)e^{i\lambda x} + D(\lambda)e^{-i\lambda x} & \text{for } \ x << 0 \end{cases}$$

where $A(\lambda), \ldots, D(\lambda)$ satisfy

$$\overline{A(\lambda)} = A(-\lambda), \ldots, \overline{D(\lambda)} = D(-\lambda) \tag{1.46}$$

and the unitarity conditions

$$|A(\lambda)|^2 + 1 = |B(\lambda)|^2, \qquad |C(\lambda)|^2 + 1 = |D(\lambda)|^2 \ . \tag{1.47}$$

The definition (1.43) of the scattering matrix gives

$$S(\lambda) = \begin{pmatrix} \frac{-A(\lambda)}{B(\lambda)C(\lambda)} & \frac{A(\lambda)}{B(\lambda)} \\ \\ \frac{B(\lambda)D(\lambda)-1}{C(\lambda)B(\lambda)} & \frac{1}{B(\lambda)} \end{pmatrix} \tag{1.48}$$

From (1.40) and Proposition 1.5(b), we find, for $\lambda \in \mathbb{R}$,

$$i\lambda\phi_-(x,\lambda) = \hat{X}(\lambda)\phi_+(x,\lambda) + \hat{Y}\phi_+(x,-\lambda) \ . \tag{1.49}$$

Using (1.49) we can express $A(\lambda), \ldots, D(\lambda)$ in terms of $\hat{X}(\lambda)$ and $\hat{Y}(\lambda)$. A simple calculation gives

$$A(\lambda) = \frac{\hat{Y}(\lambda)}{i\lambda} \ , \quad B(\lambda) = \frac{\hat{X}(\lambda)}{i\lambda} \ , \quad C(\lambda) = \frac{-\hat{Y}(\lambda)}{i\lambda} \ , \quad D(\lambda) = \frac{-\hat{Y}(\lambda)}{i\lambda} \ . \tag{1.50}$$

In the calculation, we have used the fact that $X$ and $Y$ are real and the unitarity condition $|A(\lambda)|^2 + 1 = |B(\lambda)|^2$ which implies

$$\hat{X}(\lambda)\hat{X}(-\lambda) = \lambda^2 + \hat{Y}(\lambda)\hat{Y}(-\lambda) \ . \tag{1.51}$$

Putting (1.50) into (1.48) we get (1.44). In particular, $S(\lambda)$ is meromorphic on $\mathbb{C}$ as both $X$ and $Y$ are compactly supported distributions. Suppose $S(\lambda)$ has a pole at $\lambda_0$ where Im $\lambda_0 > 0$, then $\hat{X}(\lambda)$ has a zero at $\lambda_0$, thus $B(\lambda)$ has a zero at $\lambda_0$ and $\phi_-(x,\lambda_0)$ becomes an eigenfunction of (1.41), i.e., $\lambda_0$ is a square root of the eigenvalues of $H_V$. Finally, relations (1.45) can be checked directly by using (1.44) and (1.51).

The scattering matrix $S(\lambda)$ also relates the incoming and outgoing distorted plane waves $e_\pm(x,\lambda)$, defined in Proposition 1.4.

**Proposition 1.6** *We have the following "functional equations" for the distorted plane waves* $e_\pm(x,\lambda)$

$$S(\lambda)^t J \begin{pmatrix} e_+(x,-\lambda) \\ e_-(x,-\lambda) \end{pmatrix} = \begin{pmatrix} e_+(x,\lambda) \\ e_-(x,\lambda) \end{pmatrix} \quad \textit{where} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.52}$$

**Proof.** We first recall the relations between $\phi_\pm$ and $e_\pm$ from (1.36)

$$\begin{aligned} \phi_+(x,-\lambda) &= \tfrac{1}{T(\lambda)} e_+(x,\lambda) = \tfrac{\hat{X}(\lambda)}{i\lambda}\, e_+(x,\lambda) \\ \phi_-(x,\lambda) &= \tfrac{1}{T(\lambda)} e_-(x,\lambda) = \tfrac{\hat{X}(\lambda)}{i\lambda}\, e_-(x,\lambda) \end{aligned} \tag{1.53}$$

where we have used

$$T(\lambda) = \frac{i\lambda}{\hat{X}(\lambda)} \tag{1.54}$$

which can be obtained, for example, by comparing $e_-(x,\lambda)$ and $\phi_-(x,\lambda)$ for $x >> 0$ and (1.50). For later use, we also record

$$R_\pm(\lambda) = \frac{\hat{Y}(\mp\lambda)}{\hat{X}(\lambda)} \tag{1.55}$$

which by (1.44) implies that

$$S(\lambda) = \begin{pmatrix} T(\lambda) & R_-(\lambda) \\ R_+(\lambda) & T(\lambda) \end{pmatrix} . \tag{1.56}$$

Now, putting (1.53) into (1.49) we obtain

$$i\lambda e_-(x,\lambda) = -\hat{X}(-\lambda) e_+(x,-\lambda) + \hat{Y}(\lambda) e_+(x,\lambda) . \tag{1.57}$$

Then, we compute

$$\begin{aligned} S(\lambda)^t J \begin{pmatrix} e_+(x,-\lambda) \\ e_-(x,-\lambda) \end{pmatrix} &= \frac{1}{\hat{X}(\lambda)} \begin{pmatrix} i\lambda & \hat{Y}(-\lambda) \\ \hat{Y}(\lambda) & i\lambda \end{pmatrix} \begin{pmatrix} e_-(x,-\lambda) \\ e_+(x,-\lambda) \end{pmatrix} \\ &= \frac{1}{\hat{X}(\lambda)} \begin{pmatrix} i\lambda e_-(x,-\lambda) + \hat{Y}(-\lambda) e_+(x,-\lambda) \\ \hat{Y}(\lambda) e_-(x,-\lambda) + i\lambda e_+(x,-\lambda) \end{pmatrix} \\ &= \begin{pmatrix} e_+(x,\lambda) \\ e_-(x,\lambda) \end{pmatrix} \end{aligned}$$

where we have used (1.57) in the last equality.

Our notion of incoming and outgoing behavior was motivated by the Schrödinger equation (see section 1.1) while the above definition of the scattering matrix is purely stationary. Now, we would like to connect it back to the dynamical point of view. Recall that if $H$ is a self-adjoint operator, then the initial value problem

$$\begin{cases} (i\partial_t - H)v = 0 \\ v|_{t=0} = u \end{cases} \tag{1.58}$$

is solved by the 1-parameter unitary group $e^{-itH}$, i.e.,

$$v(t) = e^{-itH}\, u \ .$$

We want to compare the free and perturbed evolutions corresponding to the self-adjoint operators $H_0$ and $H_V$ respectively. First, we want to show that for any initial data $u \in L^2(\mathbb{R})$ orthogonal to the space of eigenfunctions of $H_V$, there exist $u_\pm \in L^2(\mathbb{R})$ such that

$$e^{-itH_V}\, u \approx e^{-itH_0}\, u_\pm \quad \text{as} \quad t \to \pm\infty \ .$$

This is given by the following classical theorem whose proof does not depend on the space dimension. Hence, we present the general case in our simple setting for $V \in L^2_{\text{comp}}(\mathbb{R}^n)$.

**Theorem 1.4** *Let* $H_V = -\Delta + V$, *where* $V \in L^2_{\text{comp}}(\mathbb{R}^n)$. *If* $u \in L^2(\mathbb{R}^n)$, *the following limits exist*

$$W_\pm u = \lim_{t \to +\infty} e^{itH_V}\ e^{-itH_0}\, u \tag{1.59}$$

*Also, we have*

$$W_\pm H_0 = H_V W_\pm \tag{1.60}$$

*and*

$$\|W_\pm u\|_{L^2} = \|u\|_{L^2} \ , \tag{1.61}$$

*i.e.,* $W_\pm$ *are partial isometries intertwining the operators* $H_0$ *and* $H_V$.

**Proof.** We first prove the existence of $W_\pm$. Let

$$U(t) = e^{itH_V}\ e^{-itH_0} \ .$$

Since $e^{itH_V}$ and $e^{-itH_0}$ are unitary operators on $L^2(\mathbb{R}^n)$ which follows from the spectral theorem of self-adjoint operators, $U(t)$ is also unitary, i.e.,

$$\|U(t)w\|_{L^2(\mathbb{R}^n)} = \|w\|_{L^2(\mathbb{R}^n)} \qquad \text{for any } \ w \in L^2(\mathbb{R}^n).$$

Hence by a standard density argument, it suffices to prove the existence of limits in (1.59) for $u$ in a dense subset of $L^2(\mathbb{R}^n)$. We take

$$D = \{u \in L^2(\mathbb{R}^n) : \hat{u} \in C_0^\infty(\mathbb{R}^n \backslash \{0\})\} \ .$$

$D$ is dense in $L^2(\mathbb{R}^n)$ because $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ is dense in $L^2(\mathbb{R}^n)$ and Fourier Transform is unitary on $L^2(\mathbb{R}^n)$. Now, for $u \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \frac{d}{dt}(e^{itH_V}\, e^{-itH_0}\, u) &= ie^{itH_V}(H_V - H_0)e^{-itH_0}u \\ &= ie^{itH_V}\, V\, e^{-itH_0}u \end{aligned}$$

Thus

$$U(s)u = u + i\int_0^s e^{itH_V}\, V\, e^{-itH_0}u\, dt$$

and $W_\pm$ exist if

$$\int \|e^{itH_V}\, V\, e^{-itH_0}u\|_{L^2(\mathbb{R}^n)}\, dt = \int \|V\, e^{-itH_0}u\|_{L^2(\mathbb{R}^n)}\, dt < \infty \tag{1.62}$$

for all $u \in D$.

Take $u \in D$, then there exists $0 < r < R$ such that for $\xi \in \operatorname{supp} \hat{u}$, we have $r < |\xi| < R$. Let $u_t(x) = (e^{-itH_0}u)(x)$, then (1.62) follows if for some constant $C$ which may depend on $u$, we have

$$|u_t(x)| \leq \frac{C}{|t|^2} \tag{1.63}$$

for $x \in \operatorname{supp} V$ and for $t$ sufficiently large. To see this, we apply integration by parts to

$$\begin{aligned} u_t(x) &= e^{-itH_0}u(x) = \frac{1}{2\pi}\int_{r<|\xi|<R} e^{ix\cdot\xi - it|\xi|^2}\hat{u}(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{r<|\xi|<R}\left(\frac{1}{i(x_j - 2t\xi_j)}\,\partial_{\xi_j}\right)^2 e^{ix\cdot\xi - it|\xi|^2}\,\hat{u}(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{r<|\xi|<R} e^{ix\cdot\xi - it|\xi|^2}\left(\partial_{\xi_j}\left(\frac{1}{i(x_j - 2t\xi_j)}\right)\right)^2 \hat{u}(\xi)d\xi \ . \end{aligned}$$

We obtain (1.63) if we observe that, as $r < |\xi| < R$,

$$\left|\frac{1}{x_j - 2t\xi_j}\right| \leq \frac{C}{|t|}$$

for $x$ in some compact set and $t$ sufficiently large. As we have proved the existence of $W_\pm$, (1.61) is clear as they are strong limits of unitary operators.

Finally, (1.60) can be seen as follows. For any $s \in \mathbb{R}$,

$$e^{isH_V} W_\pm \, e^{-isH_0} = \lim_{t\to\pm\infty} e^{i(s+t)H_V} \, e^{-i(s+t)H_0} = W_\pm$$

Thus,

$$0 = \frac{1}{i} \, \partial_s \big( e^{isH_V} W_\pm \, e^{-isH_0} \big) = H_V W_\pm - W_\pm H_0$$

which is (1.60).

The operators $W_\pm$ defined in Theorem 1.4 are called wave operators. They can be defined in situations with greater generality. To illustrate this, we present the following simple example.

**Example.** Let $H_0 = D_x$, $H_V = D_x + V$, $V \in C_0^\infty(\mathbb{R})$. Then $e^{itH_0} u(x) = u(x+t)$ and since $H_V = e^{-iF} H_0 e^{iF}$ where $F' = V$, we have

$$\begin{aligned} e^{itH_V} u(x) &= e^{-iF} \, e^{itH_0} \, e^{iF} u(x) \\ &= e^{-iF(x)} \, e^{iF(x+t)} \, u(x+t) \end{aligned}$$

Thus,

$$\begin{aligned} W_\pm u(x) &= \lim_{t\to\pm\infty} e^{itH_V} \, e^{-itH_0} \, u(x) \\ &= \lim_{t\to\pm\infty} e^{-iF(x)} \, e^{iF(x)} \, e^{iF(x+t)} \, u(x) \\ &= e^{i(F(\pm\infty) - F(x))} \, u(x) \end{aligned}$$

Now, using the same wave operators, we can define the scattering operator by

$$\mathcal{S} = W_+^* W_- \tag{1.64}$$

In our simple example above, $\mathcal{S}$ is simply the multiplication operator by the constant $e^{-i\int_{\mathbb{R}} V(x)dx}$.

The following theorem relates the scattering operator to the scattering matrix defined in (1.43).

**Theorem 1.5** *The scattering operator* $\mathcal{S} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is unitary and is given by the Fourier multiplier*

$$\mathcal{S} = \Phi^* S(\cdot) \Phi \tag{1.65}$$

*where* $\Phi : L^2(\mathbb{R}) \to (L^2([0,\infty))^2$ *is defined by*

$$\Phi(u) = \begin{pmatrix} \hat{u}(\cdot) \\ \hat{u}(-\cdot) \end{pmatrix} \tag{1.66}$$

**Proof.** We first prove (1.65). Take any $v \in C_0^\infty(\mathbb{R})$, let $u = (\Phi^* S(\cdot)\Phi)v$. Thus

$$\begin{pmatrix} \hat{u}(\lambda) \\ \hat{u}(-\lambda) \end{pmatrix} = S(\lambda) \begin{pmatrix} \hat{v}(\lambda) \\ \hat{v}(-\lambda) \end{pmatrix} \tag{1.67}$$

and $u \in S(\mathbb{R})$. We need to prove $\mathcal{S}v = u$ or equivalently $W_+ u = W_- v$. Let

$$w(x) = \frac{1}{2\pi} \int_0^\infty \big[e_+(x,\lambda)\hat{v}(\lambda) + e_-(x,\lambda)\hat{v}(-\lambda)\big] d\lambda \ . \tag{1.68}$$

We claim that

$$W_- v = w \ . \tag{1.69}$$

Indeed, we have

$$\begin{aligned} e^{-itH_V} w(x) &= \frac{1}{2\pi} \int_0^\infty [e_+(x,\lambda)\hat{v}(\lambda) + e_-(x,\lambda)\hat{v}(-\lambda)]\, e^{-it\lambda^2}\, d\lambda \\ &= e^{-itH_0} v(x) + \frac{1}{2\pi} \int_0^\infty [f_+(x,\lambda)\hat{v}(\lambda) + f_-(x,\lambda)\hat{v}(-\lambda)]\, e^{-it\lambda^2}\, d\lambda \end{aligned}$$

where $f_\pm(x,\lambda) = e_\pm(x,\lambda) - e^{\pm i\lambda x} = -R_V(\lambda)(V\, e^{\pm i\lambda x})$ are outgoing and holomorphic in $\{\lambda \in C : \mathrm{Re}\ \lambda > 0,\ \mathrm{Im}\ \lambda > 0\}$ as shown in Proposition 1.4. Now, deforming the contour of integration in the last integral from $\mathbb{R}_+$ to $\Gamma_+ = \{\mu + i\mu : \mu > 0\}$, we see that the last integral tends to 0 in $L^2(\mathbb{R})$ as $t \to -\infty$ once we observe the bounds

$$\|f_\pm(\cdot\,,\lambda)\|_{L^2} \leq C\, e^{C\mathrm{Im}\ \lambda}\ , \qquad |\hat{v}(\pm\lambda)| \leq C\, e^{C\mathrm{Im}\ \lambda} \tag{1.70}$$

on $\Gamma_+$. The first estimate comes from the bound

$$\|R_V(\lambda)\|_{L^2 \to L^2} \leq \frac{1}{\mathrm{Im}^2(\lambda)} \qquad \text{on } \ \Gamma_+$$

by the Spectral Theorem and the second estimate comes from the fact that $v$ is compactly supported. Thus, our claim is proved.

Next, we let

$$\tilde{w}(x) = \frac{1}{2\pi} \int_0^\infty [e_+(x,-\lambda)\hat{u}(-\lambda) + e_-(x,-\lambda)\hat{u}(\lambda)] d\lambda \tag{1.71}$$

and we can prove by similar argument that

$$W_+ u = \tilde{w} \ .$$

Note that the sign switch comes from the fact that $e_\pm(x,-\lambda) - e^{\mp i\lambda x}$ is incoming and hence we have to deform the contour of integration into the lower half plane. we also remark that although the second estimate in (1.70) may not hold for $u$ which is only known to be Schwartz a priori, (1.71) can still be obtained by an approximation argument.

(1.65) will follow if we can prove $\tilde{w} = w$. Now

$$\begin{aligned}
\tilde{w}(x) &= \frac{1}{2\pi}\int_0^\infty (e_+(x,-\lambda), e_-(x,-\lambda)) \begin{pmatrix} \hat{u}(-\lambda) \\ \hat{u}(\lambda) \end{pmatrix} d\lambda \\
&= \frac{1}{2\pi}\int_0^\infty (e_+(x,\lambda), e_-(x,\lambda)) S(\lambda)^{-1} J \begin{pmatrix} \hat{u}(-\lambda) \\ \hat{u}(\lambda) \end{pmatrix} d\lambda \quad \text{by (1.52)} \\
&= \frac{1}{2\pi}\int_0^\infty (e_+(x,\lambda), e_-(x,\lambda)) \begin{pmatrix} \hat{v}(\lambda) \\ \hat{v}(-\lambda) \end{pmatrix} d\lambda \quad \text{by (1.67)} \\
&= w(x) \ .
\end{aligned}$$

Finally, the unitarity of $\mathcal{S}$ follows from (1.65) and the unitarity of $S(\lambda)$ and $\Phi$.

**Remark.** A different proof not specific to dimension 1 will be given in Chapter 2. In the more general context, the multiplier we used is related to the spectral decomposition of the free Laplacian in section 1.1, namely,

$$dE^0_\lambda = \Phi_0^*(\lambda)\Phi_0(\lambda) d\lambda$$

with

$$\Phi_0(\lambda) u = (\hat{u}(\lambda), \hat{u}(-\lambda)) \ .$$

Since $\mathcal{S}$ commutes with $H_0 = D_x^2$, we have formally

$$\mathcal{S} = \int_0^\infty S(\lambda) dE^0_\lambda \qquad \text{and} \qquad \mathcal{S} = \Phi_0^* S(\cdot) \Phi_0 \ .$$

As an application, we give the weak asymptotic completeness of the wave operators.

**Proposition 1.7** *We have* $\text{Ran } W_+ = \text{Ran } W_-$.

**Proof.** This is a consequence of the unitarity of $\mathcal{S}$ and can be seen clearly from the following diagram.

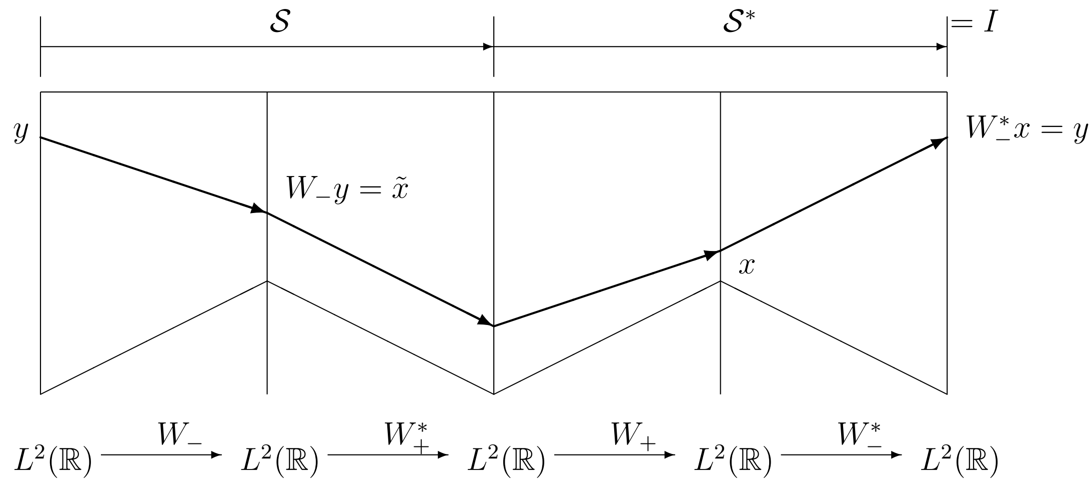


Figure 6

Take any $x \in \text{Ran } W_+$, the existence and uniqueness of $\tilde{x}$ is clear from Figure 6. Now we have

$$W_+ W_+^* \tilde{x} = x$$

which implies $\tilde{x} = x$ since $W_+$ is a partial isometry. Thus $\text{Ran } W_+ \subset \text{Ran } W_-$. Similarly, by considering $\mathcal{S}^*\mathcal{S} = I$ instead of $\mathcal{S}\mathcal{S}^* = I$, we get $\text{Ran } W_- \subset \text{Ran } W_+$.

In fact, the range of $W_+$ (or $W_-$) is characterized as the orthogonal complement of the eigenfunctions of $H_V$. More precisely,

**Proposition 1.8** *We have*

$$(\text{Ran } W_\pm)^\perp = \text{Ker } W_\pm^* = \text{Span } \{\phi_-(x, i\sqrt{-E_k})\} \tag{1.72}$$

*where the* $E_k$*'s are eigenvalues of* $H_V$*.*

**Proof.** The fact that $\phi_-(x, i\sqrt{-E_k})$ gives an eigenfunction corresponding to the eigenvalue $E_k$ is explained in the proof of Theorem 1.3. Note that all the eigenvalues of $H_V$ are simple because of the uniqueness theorem of ordinary differential equation, see Proposition 1.3.

The first equality in (1.72) is a standard fact, we only need to prove the second equality there.

First, we show $\text{Span}\{\phi_-(x, i\sqrt{-E_k})\} \subset (\text{Ran } W_\pm)^\perp$. Of course, we only need to show it for Ran $W_-$ in view of Proposition 1.7. For this, it suffices to check $\langle w(x), \phi_-(x, i\sqrt{-E_k})\rangle = 0$ for all $w(x)$ of the form

$$w(x) = \frac{1}{2\pi}\int_0^\infty [e_+(x,\lambda)\hat{v}(\lambda) + e_-(x,\lambda)\hat{v}(-\lambda)]d\lambda \qquad v \in C_0^\infty(\mathbb{R}) \ ,$$

see (1.68),(1.69). This in turn follows from, for $\lambda > 0$

$$\begin{aligned}
\langle e_\pm(x,\lambda)\phi_-(x, i\sqrt{-E_k})\rangle &= \frac{1}{E_k}\langle e_\pm(x,\lambda), H_V\phi_-(x, i\sqrt{-E_k})\rangle \\
&= \frac{1}{E_k}\langle H_V e_\pm(x,\lambda), \phi_-(x, i\sqrt{-E_k})\rangle \\
&= \frac{\lambda^2}{E_k}\langle e_\pm(x,\lambda), \phi_-(x, i\sqrt{-E_k})\rangle
\end{aligned}$$

implying $\langle e_\pm(x,\lambda), \phi_-(x, i\sqrt{-E_k})\rangle = 0$ as $\frac{\lambda^2}{E_k} \neq 1$. It remains to show $(\text{Ran } W_-)^\perp \subset \text{Span}\{\phi_-(x, i\sqrt{-E_k})\}$ or equivalently $(\text{Span}\{\phi_-(x, i\sqrt{-E_k}\})^\perp \subset \text{Ran } W_-$. By the spectral decomposition of $H_V$, Theorem 1.2, a generic element of $(\text{Span}\{\phi_-(x, i\sqrt{-E_k}\})^\perp$ is of the form

$$\tilde{w}(x) = \frac{1}{2\pi}\int_0^\infty [e_+(x,\lambda)\tilde{f}_+(\lambda) + e_-(x,\lambda)\tilde{f}_-(\lambda)]\, d\lambda$$

where

$$\tilde{f}_\pm(\lambda) = \int_{-\infty}^\infty \overline{e_\pm(y,\lambda)} f(y) dy \ , \qquad f \in C_0^\infty(\mathbb{R}) \ .$$

Note that $\tilde{f}_\pm(\lambda) \in L^2(\mathbb{R}_+)$ as $e_\pm(y,\lambda)$ is uniformly bounded by (1.37). We can then find $g \in L^2(\mathbb{R})$ such that $\hat{g}(\lambda) = \tilde{f}_+(\lambda)$ and $\hat{g}(-\lambda) = \tilde{f}_-(\lambda)$ for $\lambda > 0$. Thus $\tilde{w}(x) \in \text{Ran } W_-$ and this completes the proof of Proposition 1.8.

## 1.4 Resonances

We have seen in Theorem 1.2 that the poles of the outgoing resolvent $R_V(\lambda)$ in the upper half plane $\{\text{Im } \lambda > 0\}$ corresponds to the eigenvalues of $H_V$. In Theorem 1.1, we saw that $R_V(\lambda)$ as operator

$$R_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R})$$

has a meromorphic continuation to $\mathbb{C}$. Its poles are some most important objects in scattering theory and are called resonances or scattering poles. We will provide more motivation and justification later. We start with some preliminaries on multiplicity.

**Proposition 1.9** *If the multiplicity of a pole* $\lambda_0 \neq 0$ *of* $R_V(\lambda)$ *is defined by*

$$m_R(\lambda_0) = \text{rank}\, \frac{1}{2\pi i} \oint_{\lambda_0} R_V(\lambda) d\lambda \ , \tag{1.73}$$

*then*

$$\begin{aligned} m_R(\lambda_0) &= \frac{1}{2\pi i} \oint \frac{\hat{X}'(\lambda)}{\hat{X}(\lambda)}\, d\lambda \\ &= \textit{the order of vanishing of } \hat{X}(\lambda) \textit{ at } \lambda_0 \end{aligned} \tag{1.74}$$

*where* $X \in \mathcal{E}'(\mathbb{R})$ *is defined in Proposition 1.5.*

**Proof.** Recall that $\phi_\pm(x,\lambda)$ are a pair of linearly independent solutions to $(H_V - \lambda^2)u = 0$ defined in Lemma 1.1.

Apply (1.25) to the operator $H_V - \lambda^2$ and use $\psi_1 = \phi_+(x,-\lambda)$ and $\psi_2 = \phi_-(x,-\lambda)$, we get

$$R_V(\lambda)(x,y) = \frac{1}{2\hat{X}(\lambda)} (\phi_+(x,-\lambda)\phi_-(x,\lambda)(x-y)^0_+ + \phi_+(y,-\lambda)\phi_-(x,\lambda)(x-y)^0_-) \ . \tag{1.75}$$

Using (1.49), namely

$$i\lambda\phi_+(x,\lambda) = \hat{X}(\lambda)\phi_+(x,\lambda) + \hat{Y}(\lambda)\phi_+(x,-\lambda)$$

we obtain from (1.75) and (1.73) that

$$m_R(\lambda_0) = \text{rank Res}_{\lambda_0} \left( \frac{\hat{Y}(\lambda)}{2i\lambda\hat{X}(\lambda)}\, \phi_+(\cdot,-\lambda) \otimes \phi_+(\cdot,\lambda) \right) \ .$$

Now suppose the order of vanishing for $\hat{X}(\lambda)$ at $\lambda_0$ is $k+1$, then the residue of

$$\frac{\hat{Y}(\lambda)}{2i\lambda\hat{X}(\lambda)}\, \phi_+(\cdot,-\lambda) \otimes \phi_+(\cdot,\lambda)$$

at $\lambda_0$ is given by, as $\hat{Y}(\cdot)$ and $\phi_+(x,\cdot)$ are entire,

$$\begin{aligned} &\frac{1}{2ik!}\, \partial_\lambda^k \left( \frac{\hat{Y}(\lambda)}{\lambda}\, \phi_+(\cdot,-\lambda) \otimes \phi_+(\cdot,\lambda) \right) \Bigg|_{\lambda=\lambda_0} \\ &= \left[ \sum_{\ell=0}^{k} \partial_\lambda^\ell \phi_+(\cdot,-\lambda) \otimes \left( \sum_{j_1,j_2 \geq 0, j_1+j_2+\ell=k} c_{j_1,j_2,\ell}\, \partial_\lambda^{j_1} \left( \frac{\hat{Y}(\lambda)}{\lambda} \right) \partial_\lambda^{j_2} \phi_+(\cdot,-\lambda) \right) \right]_{\lambda=\lambda_0} \end{aligned}$$

for some nonzero constants $c_{j_1,j_2,\ell}$. The above operator is of rank $k+1$ because

(i) $\partial_\lambda^j \phi_+(x,-\lambda)|_{\lambda=\lambda_0}$, $j = 0,1,2,\dots k$, are linearly independent functions in $x$ since $\partial_\lambda^j \phi_+(x,-\lambda)|_{\lambda=\lambda_0} = (ix)^j\, e^{i\lambda_0 x}$ for $x >> 0$

and

(ii) the coefficient of $\partial_\lambda^j \phi_+(x,-\lambda)|_{\lambda=\lambda_0}$, $j = 0,\dots k$, since $\hat{Y}(\lambda_0) \neq 0$ by the unitarity relation (1.51).

This finishes the proof of Proposition 1.9.

Theorem 1.3 shows that the poles of the resolvent coincides with the poles of the scattering matrix. For matrix-valued meromorphic function (or more generally, for operator valued meromorphic function), the natural notion of multiplicity of poles is given by

$$m_S(\lambda_0) = -\frac{1}{2\pi i}\oint_{\lambda_0} \mathrm{tr}(S(\lambda)^{-1}S'(\lambda))d\lambda \tag{1.76}$$

which is the same as the order of the poles of det $S(\lambda)$ because

$$\frac{(\det\, S(\lambda))'}{\det\, S(\lambda)} = \mathrm{tr}(S(\lambda)^{-1}S'(\lambda)) \ . \tag{1.77}$$

**Proposition 1.10** *The definition of multiplicities given by* (1.73) *and* (1.76) *are related by*

$$m_S(\lambda) = m_R(\lambda) - m_R(-\lambda) \tag{1.78}$$

**Proof.** Using (1.44), we compute

$$\det\, S(\lambda) = \frac{-X(-\lambda)}{\hat{X}(\lambda)} \tag{1.79}$$

Proposition 1.10 then follows from Proposition 1.9.

**Remark.** If $\mathrm{Im}\ \lambda < 0$, $m_R(\lambda) \neq 0$ only for $\lambda^2$ being an eigenvalue of $H_V$. With a slight abuse of terminology, we say that the poles of $S(\lambda)$ and $R_V(\lambda)$ coincide with multiplicities.

We will now give some motivations for the study of resonances. We start with the time delay operator. For simplicity, write $\chi_r(x) = 1\!\!1_{\{|x|<r\}}$. For $f \in C_0^\infty(\mathbb{R})$, put

$$S_r(f) = \int_{-\infty}^{\infty} \|\chi_r\, e^{-itH_V}\, W_- f\|^2\, dt$$

and

$$S_r^0(f) = \int_{-\infty}^{\infty} \|\chi_r\, e^{-itH_0}\, f\|^2\, dt$$

where $W_-$ is one of the wave operators studied in section 1.3.

Define an operator $\tilde{T}_r$ by its corresponding quadratic form

$$\langle \tilde{T}_r f, f\rangle = S_r(f) - S_r^0(f) \ ,$$

then the time delay operator $\tilde{T}$ is given by

$$\langle \tilde{T} f, f\rangle = \lim_{r\to\infty} \langle \tilde{T}_r f, f\rangle \ . \tag{1.80}$$

The next proposition guarantees the existence of the limit. Here we note that, by definition of $W_-$,

$$e^{-itH_V} W_- f \sim e^{-itH_0} f \quad \text{as} \quad t \to -\infty \ .$$

Hence $W_- f$ and $f$ evolve the same way under the perturbed and free propagation respectively for large negative times.

**Proposition 1.11** (Eisenbud-Wigner Formula) *The operator* $\tilde{T}$ *given by* (1.80) *exists and*

$$\tilde{T} = \Phi^* T(\cdot)\Phi \tag{1.81}$$

*where* $T(\lambda) = -2\lambda i S(\lambda)^* \frac{d}{d\lambda} S(\lambda)$ *and* $\Phi$ *is as in Theorem* 1.5.

**Outline of proof.** (1.81) is equivalent to, for $f \in C_0^\infty(\mathbb{R})$

$$\begin{pmatrix} \widehat{\tilde{T}f}(\lambda) \\ \widehat{\tilde{T}f}(-\lambda) \end{pmatrix} = -2i\lambda S(\lambda)^* S'(\lambda) \begin{pmatrix} \hat{f}(\lambda) \\ \hat{f}(-\lambda) \end{pmatrix} \tag{1.82}$$

which in turn is equivalent to

$$\begin{aligned}
\langle \tilde{T} f, f\rangle &= \int_{-\infty}^{\infty} (\tilde{T} f)(x) \bar{f}(x) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\tilde{T}f}(\lambda) \overline{\hat{f}(\lambda)}\, d\lambda \quad \text{by Plancherel formula} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\widehat{\tilde{T}f}(\lambda) \overline{\hat{f}(\lambda)} + \widehat{\tilde{T}f}(-\lambda) \overline{\hat{f}(-\lambda)}) d\lambda \qquad (1.83) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\bar{\hat{f}}(\lambda), \bar{\hat{f}}(-\lambda)) \begin{pmatrix} \widehat{\tilde{T}f}(\lambda) \\ \widehat{\tilde{T}f}(-\lambda) \end{pmatrix} d\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\bar{\hat{f}}(\lambda), \bar{\hat{f}}(-\lambda)) (-2i\lambda S(\lambda)^* S'(\lambda)) \begin{pmatrix} \hat{f}(\lambda) \\ \hat{f}(-\lambda) \end{pmatrix} d\lambda
\end{aligned}$$

for $f \in C_0^\infty(\mathbb{R})$. Now, using the expression of $S(\lambda)$ in terms of the transmission and reflection coefficients given in (1.56), we get

$$\begin{aligned} S(\lambda)^* S'(\lambda) &= \begin{pmatrix} T(-\lambda)T'(\lambda) + R_+(-\lambda)R'_+(\lambda) & T(-\lambda)R'_-(\lambda) + R_+(-\lambda)T'(\lambda) \\ R_-(-\lambda)T'(\lambda) + T(-\lambda)R'_+(\lambda) & R_-(-\lambda)R'_-(\lambda) + T(-\lambda)T'(\lambda) \end{pmatrix} \\ &:= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned} \tag{1.84}$$

Thus, we need to prove that $\lim_{r\to\infty} (S_r(f) - S_r^0(f))$ exists and

$$\begin{aligned} &\lim_{r\to\infty} (S_r(f) - S_r^0(f)) \\ &= \frac{1}{2\pi}\int_0^\infty (-2i\lambda)\Big[a\hat{f}(\lambda)\overline{\hat{f}}(\lambda) + d\hat{f}(-\lambda)\overline{\hat{f}}(-\lambda) + b\hat{f}(-\lambda)\overline{\hat{f}}(\lambda) + c\hat{f}(\lambda)\overline{\hat{f}}(-\lambda)\Big]d\lambda\ . \end{aligned} \tag{1.85}$$

To compute $S_r(f) - S_r^0(f)$, we use the formula for $W_-$ given by (1.68) (and (1.69)), we get

$$\begin{aligned} S_r(f) &= \int_{-\infty}^{\infty} \|\chi_r\, e^{-itH_V} W_- f\|^2 dt \\ &= \int_{-\infty}^{\infty} \|\chi_r\, W_- e^{-itH_0} f\|^2 dt \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-r}^{r} |\int_0^\infty (e_+(x,\xi)\widehat{e^{-itH_0}f}(\xi) + e_-(x,\xi)\widehat{e^{-itH_0}f}(-\xi))d\xi|^2\, dxdt \\ &\qquad \text{by (1.68)} \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-r}^{r} |\int_0^\infty e^{-it\xi^2}(e_+(x,\xi)\hat{f}(\xi) + e_-(x,\xi)\hat{f}(-\xi))d\xi|^2\, dxdt \\ &= \left(\frac{1}{2\pi}\right)^3 \int_{-r}^{r}\int_{-\infty}^{\infty} |\int_0^\infty \mathcal{F}_{t\mapsto\mu}(e^{-it\xi^2}(e_+(x,\xi)\hat{f}(\xi) + e_-(x,\xi)\hat{f}(-\xi))d\xi|^2\, d\mu dx \\ &\qquad \text{by Plancherel formula} \\ &= \frac{1}{2\pi}\int_{-r}^{r}\int_{-\infty}^{\infty} |\int_0^\infty \delta_0(\mu+\xi^2)(e_+(x,\xi)\hat{f}(\xi) + e_-(x,\xi)\hat{f}(-\xi))d\xi|^2\, d\mu dx \\ &= \frac{1}{2\pi}\int_{-r}^{r}\int_0^\infty |e_+(x,\sqrt{\mu})\hat{f}(\sqrt{\mu}) + e_-(x,\sqrt{\mu})\hat{f}(-\sqrt{\mu})|^2 d\mu dx \\ &= \frac{1}{2\pi}\int_{-r}^{r}\int_0^\infty 2\lambda|e_+(x,\lambda)\hat{f}(\lambda) + e_-(x,\lambda)\hat{f}(-\lambda)|^2 d\lambda dx \\ &= \frac{1}{2\pi}\int_0^\infty\int_{-r}^{r} 2\lambda(e_+(x,\lambda)e_+(x,-\lambda)\hat{f}(\lambda)\overline{\hat{f}(\lambda)} + e_-(x,\lambda)e_-(x,-\lambda)\hat{f}(-\lambda)\overline{\hat{f}(-\lambda)} \\ &\qquad + e_-(x,\lambda)e_+(x,-\lambda)\hat{f}(-\lambda)\overline{\hat{f}(\lambda)} + e_+(x,\lambda)e_-(x,-\lambda)\hat{f}(\lambda)\overline{\hat{f}(-\lambda)})dxd\lambda \end{aligned} \tag{1.86}$$

Similarly, we have

$$S_r^0(f) = \int_{-\infty}^{\infty} \|\chi_r\, e^{-itH_0} f\|^2 dt \tag{1.87}$$
$$= \frac{1}{2\pi}\int_0^\infty \int_{-r}^{r} 2\lambda(\hat f(\lambda)\overline{\hat f(\lambda)} + \hat f(-\lambda)\overline{\hat f(-\lambda)} + e^{-i\lambda x}\hat f(-\lambda)\overline{\hat f(\lambda)} + e^{i\lambda x}\hat f(\lambda)\overline{\hat f(-\lambda)})dxd\lambda$$

Thus, to prove (1.85), we need

$$\lim_{r\to\infty}\int_0^\infty \int_{-r}^{r} 2\lambda(e_\pm(x,\lambda)e_\pm(x,-\lambda) - 1)dx\, \hat f(\pm\lambda)\overline{\hat f}(\pm\lambda)d\lambda$$
$$= \int_0^\infty (-2i\lambda)[T(-\lambda)T'(\lambda) + R_\pm(-\lambda)R'_\pm(-\lambda)]\hat f(\pm\lambda)\overline{\hat f}(\pm\lambda)d\lambda$$

and

$$\lim_{r\to\infty}\int_0^\infty \int_{-r}^{r} 2\lambda(e_\pm(x,\lambda)e_\mp(x,-\lambda) - e^{\pm i\lambda x})dx\, \hat f(\pm\lambda)\overline{\hat f}(\mp\lambda)d\lambda$$
$$= \int_0^\infty (-2i\lambda)[T(-\lambda)R'_\pm(\lambda) + R_\mp(-\lambda)T'(\lambda)]\hat f(\pm\lambda)\overline{\hat f}(\mp\lambda)d\lambda$$

These can be proved by exactly the same type of computation as we prove the Birman Krein formula in section 1.5. Since we are going to give a detailed exposition there, we will not carry out the computation here. Indeed, the first equality above follows directly from the computation there.

This finishes our outline of the proof of Proposition 1.11.

Another motivation for the study of resonances is the fact that in a weaker sense, resonances replace eigenvalues in expansion with modes (eigenfunctions). We recall that if we have $H_V = D_x^2 + V$ on $[a,b]$ with Dirichlet (or Neumann) boundary condition, then the problem

$$\begin{cases} (H_V - \lambda^2)u = 0 & \text{on } (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

has a distinct set of solutions $(i\sqrt{-E_k},\, v_k), (\lambda_j, u_j)$ with $E_N < \cdots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty$, $\int_a^b |u_j|^2 dx = \int_a^b |v_k|^2 dx = 1$. If we consider the wave equation

$$\begin{cases} (D_t^2 - H_V)w = 0 & \text{on } \mathbb{R}\times(a,b) \\ w(0,x) = w_0(x) & \text{on } [a,b] \\ \partial_t w(0,x) = w_1(x) & \text{on } [a,b] \\ w(t,a) = w(t,b) = 0 & \text{on } \mathbb{R}, \end{cases}$$

then

$$
\begin{aligned}
w(t,x) &= \sum_{k=1}^{N} \cosh(t\sqrt{-E_k}) a_k v_k(x) + \sum_{k=1}^{N} \sinh(t\sqrt{-E_k}) b_k v_k(x) \\
&\quad + \sum_{j=0}^{\infty} \cos(t\lambda_j) c_j u_j(x) + \sum_{j=0}^{\infty} \sin(t\lambda_j) d_j u_j(x)
\end{aligned} \tag{1.88}
$$

where

$$
\begin{aligned}
a_k &= \int_a^b w_0(x)\bar{v}_k(x)dx \ , \qquad b_k = \int_a^b w_1(x)\bar{v}_k(x)dx \ , \\
c_j &= \int_a^b w_0(x)\bar{u}_j(x)dx \ , \qquad d_j = \int_a^b w_1(x)\bar{u}_j(x)dx \ .
\end{aligned}
$$

We now give an analogue of (1.88) for problems on open domains involving resonances. In its proof we need the following

**Lemma 1.2** *Suppose $V \in L^\infty_{\text{comp}}(\mathbb{R})$. Then for any $\rho \in C_0^\infty(\mathbb{R})$ satisfying $\rho V = V$, there are constants $A', C, T$ depending on the support of $\rho$ such that*

$$
\|\rho R_V(\lambda)\rho\|_{L^2\to L^2} \leq \frac{C}{|\lambda|}\, e^{T|\operatorname{Im}\lambda|} \tag{1.89}
$$

*for $\operatorname{Im}\lambda \geq -A' - \delta\log\langle\lambda\rangle$ and $\lambda$ sufficiently large. Here $\delta$ is a constant depending only on the support of $V$. In particular there are only finitely many resonances in the region*

$$
\{\operatorname{Im}\lambda \geq -A - \log\langle\lambda\rangle\}
$$

*for any $A$.*

**Proof.** First, note the following obvious estimate of the free resolvent

$$
\|\rho R_0(\lambda)\rho\|_{L^2\to L^2} \leq \frac{1}{|\lambda|}\, e^{T|\operatorname{Im}\lambda|} \tag{1.90}
$$

for some constant $T$ depending on the support of $\rho$, see (1.9). Since we have, from (1.20),

$$
\rho R_V(\lambda)\rho = \rho R_0(\lambda)\rho(I + V R_0(\lambda)\rho_1)^{-1}
$$

where $\rho_1 \in C_0^\infty(\mathbb{R})$ is any function satisfying $\rho\rho_1 = \rho_1$, (1.89) holds in the region where

$$
\|V R_0(\lambda)\rho_1\|_{L^2\to L^2} \leq \tfrac{1}{2} \ .
$$

Our lemma clearly follows from (1.90). Note that the constant $\delta$ does not depend on $\rho$ as we can choose $\rho_1$ with support as close to that of $V$ as we like.

**Remark.** Here, we draw a consequence of Lemma 1.2. Since

$$(D_x^2 + V - \lambda^2)\rho R_V(\lambda)\rho = \rho^2 I + [D_x^2, \rho] R_V(\lambda)\rho$$

we have

$$D_x^2(\rho R_V(\lambda)\rho) = \rho^2 I + (D_x^2\rho + 2D_x\rho \cdot D_x)R_V(\lambda)\rho - V\rho R_V(\lambda)\rho + \lambda^2 \rho R_V(\lambda)) \,.$$

Thus

$$\begin{aligned} \|\rho R_V(\lambda)\rho\|_{L^2\to H^2} &\leq C\|D_x^2 \rho R_V(\lambda)\rho\|_{L^2\to L^2} \\ &\leq C\Big(1 + \|\rho_1 R_V(\lambda)\rho\|_{L^2\to L^2} + \|D_x\rho\|_{L^\infty}\|\rho_1 R_V(\lambda)\rho\|_{L^2\to H^1} \\ &\qquad + (1+\lambda^2)\|\rho R_V(\lambda)\rho\|_{L^2\to L^2}\Big) \end{aligned}$$

where $\rho_1 \in C_0^\infty(\mathbb{R})$ with $\rho_1\rho = \rho$. If $\|D_x\rho\|_{L^\infty}$ is small, we have for large $\lambda$,

$$\|\rho R_V(\lambda)\rho\|_{L^2\to H^2} \leq C|\lambda|\, e^{T'|\mathrm{Im}\,\lambda|} \tag{1.91}$$

in the region $\mathrm{Im}\,\lambda \geq -A' - \delta\log\langle\lambda\rangle$.

Now we can state the analogue of (1.88).

**Theorem 1.6** *Suppose $w(t,x)$ is the solution of*

$$\begin{cases} (D_t^2 - H_V)w(t,x) = 0 & on\ \mathbb{R}\times\mathbb{R} \\ w(0,x) = w_0(x) & on\ \mathbb{R} \\ \partial_t w(0,x) = w_1(x) & on\ \mathbb{R}, \end{cases} \tag{1.92}$$

*where $w_0 \in H^1_{\mathrm{comp}}(\mathbb{R})$, $w_1 \in L^2_{\mathrm{comp}}(\mathbb{R})$ with* $\mathrm{supp}\, w_0$, $\mathrm{supp}\, w_1 \subset \{|x| < R\}$. *Then, for any $A > 0$,*

$$\begin{aligned} w(t,x) &= \sum_{\mathrm{Im}\,\lambda>0} \mathrm{Res}[(iR_V(\lambda)w_1 + \lambda R_V(\lambda)w_0)e^{-i\lambda t}] \\ &\quad + \sum_{0<-\mathrm{Im}\,\lambda\leq A+\delta\log\langle\lambda\rangle} \mathrm{Res}[(iR_V(\lambda)w_1 + \lambda R_V(\lambda)w_0)e^{-i\lambda t}] + E_A(t) \end{aligned} \tag{1.93}$$

*where $E_A(t)$ satisfies the estimate*

$$\|1\!\!1_{\{|x|<K\}} E_A(t)\| \leq C_{K,R} e^{-(A-\varepsilon)(t-T')}(\|w_0\|_{H^1} + \|w_1\|_{L^2}) \tag{1.94}$$

*for any $\varepsilon > 0$ and some constants $T', K$ sufficiently large.*

**Remarks.** 1. For any $A > 0$, the sum in (1.93) is finite because of Lemma 1.2.

2. The first term on the right-hand side of (1.93) corresponds to the eigenvalues of $H_V$ and can be written as

$$\sum_{k=1}^{N}(a_k \cosh t\sqrt{-E_k}\,\phi_-(x,i\sqrt{-E_k}) + b_k \sinh t\sqrt{-E_k}\,\phi_-(x,i\sqrt{-E_k}))$$

where $E_k$, $k = 1,\dots N$ are the eigenvalues of $H_V$ and $\phi_-(x,i\sqrt{-E_k})$ the corresponding eigenfunctions as explained in Theorem 1.3, the $a_k$'s, $b_k$'s are given by

$$\begin{aligned} a_k &= \frac{1}{\|\phi_-(x,i\sqrt{-E_k})\|_{L^2}} \int w_0(x)\phi_-(x,i\sqrt{-E_k})dx \\ b_k &= \frac{1}{\|\phi_-(x,i\sqrt{-E_k})\|_{L^2}} \int w_1(x)\phi_-(x,i\sqrt{-E_k})dx \end{aligned}$$

**Proof of Theorem 1.6**. For simplicity, we assume that $H_V$ has no negative eigenvalues as their contribution to (1.93) is clear. Also, we will only consider (1.92) with $w_0 \equiv 0$ as the proof below clearly works in the case $w_1 \equiv 0$ if we replace $\frac{\sin t\lambda}{\lambda}$ by $\cos t\lambda$ in the formula for $w(t,x)$. The general case is then obtained by taking linear combinations.

With the above simplications understood, by the Spectral Theorem, the solution of (1.92) can be written as

$$w(t) = \int_0^\infty \frac{\sin t\lambda}{\lambda}\, dE_\lambda(w_1)$$

Using Stone's Formula to write $dE_\lambda$ in terms of $R_V(\lambda)$, we get

$$\begin{aligned} w(t) &= \frac{1}{\pi i}\int_0^\infty \sin t\lambda (R_V(\lambda) - R_V(-\lambda))w_1 d\lambda \\ &= \frac{1}{\pi i}\int_0^\infty \frac{e^{it\lambda} - e^{-it\lambda}}{2i}(R_V(\lambda) - R_V(-\lambda))w_1 d\lambda \\ &= \frac{1}{\pi i}\left(\frac{1}{i}\right)\left[\int_{-\infty}^\infty e^{it\lambda}R_V(\lambda))w_1 d\lambda - \int_{-\infty}^\infty e^{-it\lambda}R_V(\lambda))w_1 d\lambda\right] \end{aligned} \tag{1.95}$$

Now, as $R_V(\lambda)$ is holomorphic in the upper half-plane, we can deform the contour of integration of the first term on the right-hand side of (1.95) to the contour illustrated in Figure 7.

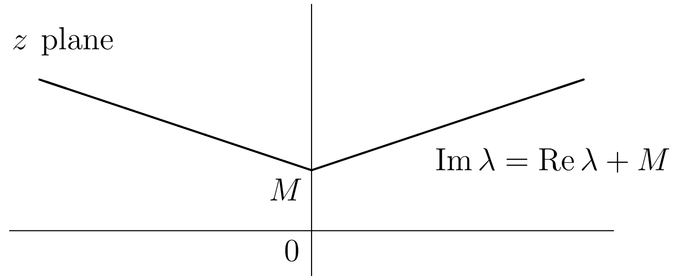


Figure 7

By letting $M \to \infty$, we can eliminate it from (1.95).

Next, for $K$ large enough, we can choose $\rho \in C_0^\infty(\mathbb{R})$ with $\rho V = V$, $\rho 1\!\!1_{\{|x|<K\}} = \rho$, $\rho 1\!\!1_{\{|x|<K\}} = 1\!\!1_{\{|x|<K\}}$ such that (1.89) holds, we then have

$$\rho w(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-it\lambda} \rho(iR_V(\lambda))\rho w_1 d\lambda$$

By (1.91), we can deform the contour of integration to the contour illustrated in Figure 8

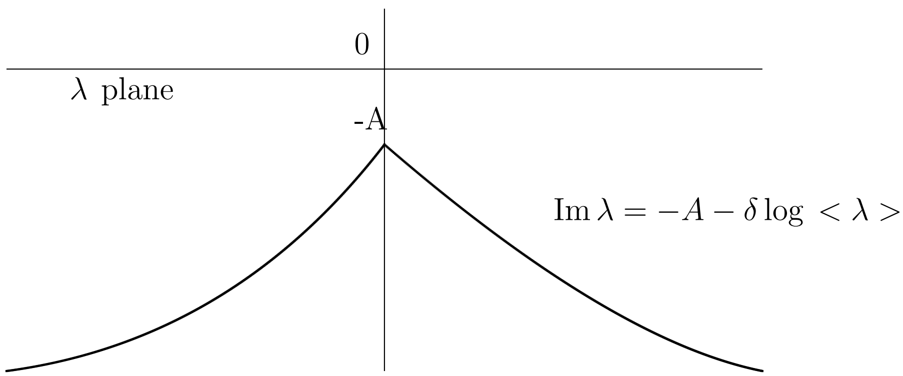


Figure 8

and we have

$$\rho w(t) = \sum_{0<\operatorname{Im}\lambda \leq -A+\delta \log\langle\lambda\rangle} \operatorname{Res}((i\rho R_V(\lambda)w_1)e^{-i\lambda t}) + E_{A,\rho}(t)$$

with

$$\|E_{A,\rho}(t)\|_{H^1} \leq C_\rho e^{-(A-\varepsilon)(t-T')}(\|w_1\|_{L^2})$$

Letting $K$ and support of $\rho$ go to infinity, we obtain (1.93) and (1.94).

Comparison of Theorem 1.6 with the normal mode expansion (1.88) shows that resonances are the natural analogue of eigenvalues for scattering problems. Now, it is classical that for $H_V$ on $[a,b]$, with either the Dirichlet or Neumann boundary conditions, we have

$$\#\{\lambda_j : \lambda_j \leq r\} = \frac{b-a}{\pi} r + O(1) \ .$$

There is also an analogous result for resonances.

**Theorem 1.7** *Let* $m_R(\lambda)$ *be the multiplicity of resonances given by* (1.73)*, then*

$$\sum_{|\lambda| \leq r} m_R(\lambda) = \frac{2|\text{ch supp}V|}{\pi}(r + o(1)) \tag{1.96}$$

*where* ch supp$V$ *is the convex hull of the support of* $V$.

**Proof.** By Proposition 1.9, Theorem 1.7 is a statement about the distribution of zeros of the entire function $\hat{X}(\lambda)$. To prove it, we need the following generalization to distributions of a classical result of Titchmarch.

**Lemma 1.3** *If* $u \in \mathcal{E}'(\mathbb{R})$*, then*

$$N_{\hat{u}}(r) = \frac{|\text{ch supp}u|}{\pi}(r + o(1)) \tag{1.97}$$

*where*

$$N_f(r) = \sum_{|z| \leq r} \frac{1}{2\pi i} \oint_z \frac{f'(w)}{f(w)} \, dw$$

*is the counting function of the zeros of* $f$.

**Proof.** Recall that the classical Theorem of Titchmarch gives Lemma 1.3 when $u \in L^1_{\text{comp}}(\mathbb{R})$. We need to extend it to $u \in \mathcal{E}'(\mathbb{R})$. First, observe that Titchmarch's Theorem implies that for $u, v \in L^1_{\text{comp}}(\mathbb{R})$, we have

$$\text{ch supp}(u * v) = \text{ch supp}\, u + \text{ch supp}\, v \ . \tag{1.98}$$

In fact, since $\widehat{u * v} = \hat{u}\hat{v}$, (1.97) implies

$$|\text{ch supp}\ u * v| = |\text{ch supp}\, u| + |\text{ch supp}\, v|$$

and we easily have

$$\text{ch supp}\ u * v \subset \text{ch supp}\, u + \text{ch supp}\, v \ .$$

Thus, (1.98) follows. Now (1.98) can be generalized easily to compactly supported distributions. To see this, let $u, v \in \mathcal{E}'(\mathbb{R})$ and $\phi_\varepsilon \in C_0^\infty(\mathbb{R})$ be supported in $(-\varepsilon, \varepsilon)$. Then

$$\begin{aligned} &\text{ch supp}(u * \phi_\varepsilon) + \text{ch supp}(v * \phi_\varepsilon) \\ &\quad = \text{ch supp}((u * v) * (\phi_\varepsilon * \phi_\varepsilon)) \subset \text{ch supp } u * v + (-2\varepsilon, 2\varepsilon)\ . \end{aligned}$$

By letting $\phi_\varepsilon \to \delta_0$ as $\varepsilon \to 0$, we obtain

$$\text{ch supp } u + \text{ch supp}\, v \subset \text{ch supp}(u * v)\ .$$

As the reverse inclusion is clearly true, we obtain (1.98) for $u, v \in \mathcal{E}'(\mathbb{R})$.

Now, to see (1.97) for $u \in \mathcal{E}'(\mathbb{R})$, we apply Titchmarch's Theorem to $\phi \in C_0^\infty(\mathbb{R})$ and $u * \phi \in C_0^\infty(\mathbb{R})$, so that

$$\frac{\pi}{r} N_{\hat{u}}(r) = \frac{\pi}{r}\big(N_{\hat{u}\hat{\phi}}(r) - N_{\hat{\phi}}(r)\big) \sim |\text{ch supp } u * \phi| - |\text{ch supp}\, \phi| = |\text{ch supp } u|$$

where we have used (1.98) in the last equality.

Going back to the proof of Theorem 1.7, it is now clear that it suffices to show that

$$\text{ch supp } X = [-2(b-a), 0] \quad \text{where} \quad [a, b] = \text{ch supp } V\ . \tag{1.99}$$

Assume the contrary, that is, for some $\varepsilon_-, \varepsilon_+ \geq 0$, $\varepsilon_- + \varepsilon_+ > 0$, we have

$$\text{ch supp } X = [-2(b-a) + \varepsilon_-, \varepsilon_+]\ .$$

We first need the following

**Lemma 1.4** *Suppose that* $V \in L^\infty_{\text{comp}}(\mathbb{R})$. *Then* $X - \delta_0'(x) \in C([-2(b-a), 0])$.

That immediately shows that $\varepsilon_+ = 0$ and to obtain a contradictions we assume that $\varepsilon_- > 0$. Recall the unitarity relation,

$$\hat{X}(\lambda)\hat{X}(-\lambda) = \lambda^2 + \hat{Y}(\lambda)\hat{Y}(\lambda)\,.$$

The density of zeros of the left hand side is given by $2c/\pi$, where $c = 2(b-a) - 2\epsilon_-$. Since the convex hull of $YY(-\bullet) - \delta''$ is the same as $\text{ch supp}\, Y - \text{ch supp}\, Y$ it follows that

$$\text{ch}\, Y = [d_-, d_+] \neq [2a, 2b]\,.$$

Suppose that $d_- > a$. Then $\partial_y A_-(x, y)$ must vanish in $|x - d_-/2| < d_-/2 - y$ by causality. But that means that $V(x)\delta(x-y) = 0$ for $x < d_-/2 > a$ which contradicts $\text{ch supp} V = [a, b]$. A similar argument using $A_+$ shows that $d_+ = 2b$.

Thus, our proof of Theorem 1.7 is completed.

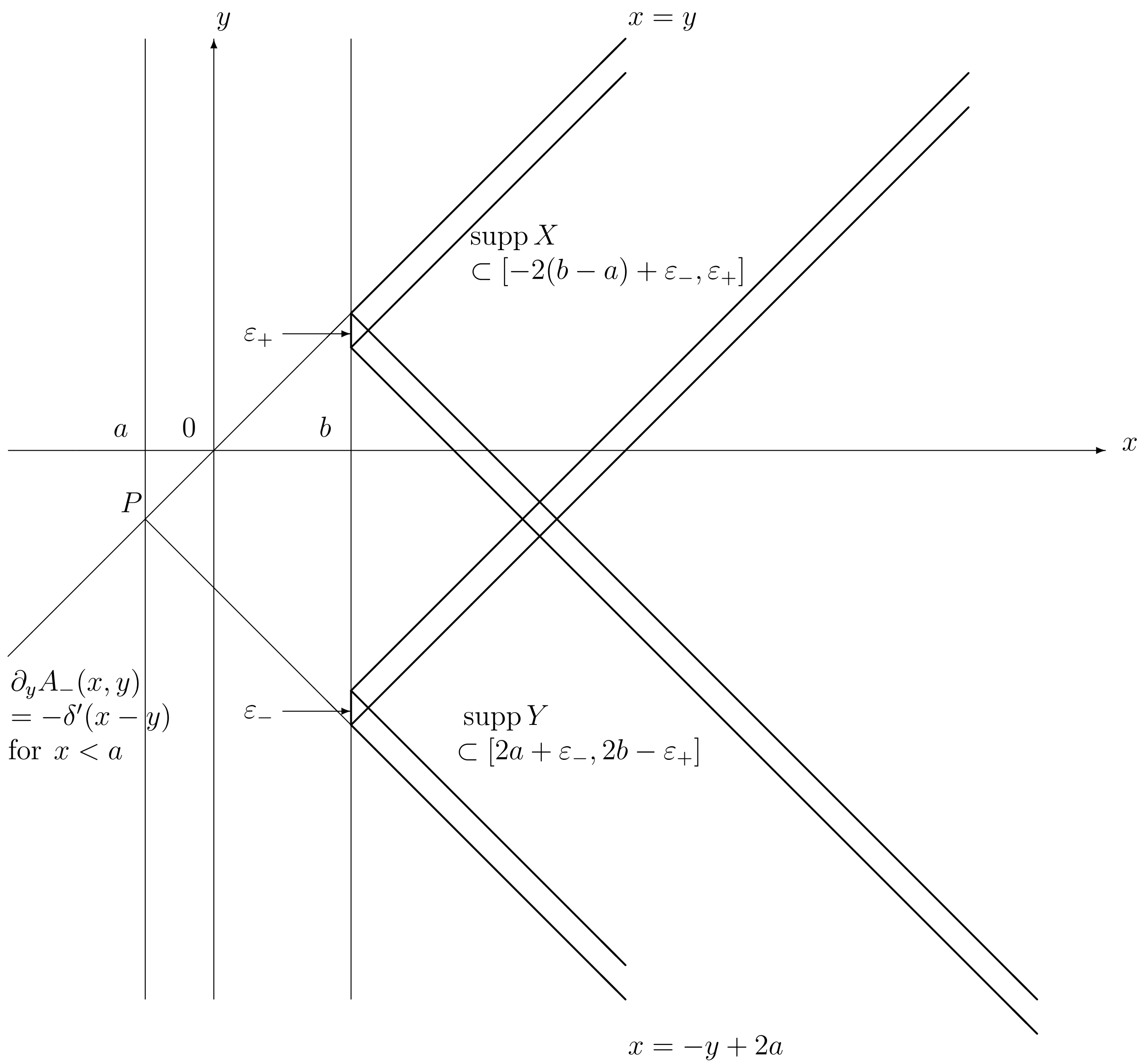


Figure 9

Finally, we discuss an important characterization of resonances which comes from the method of complex scaling. It is particularly clear in the case of compactly supportd potential on $\mathbb{R}$. To introduce it, we first review the restriction of holomorphic differential operator on $\mathbb{C}$ to smooth curves.

Let $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. If $\Gamma \subset \mathbb{C}$ is a smooth curve and $u \in C^\infty(\Gamma)$, we define

$$(\partial_z|_\Gamma)u = (z'(t))^{-1}\partial_t u$$

where $\Gamma = \{z(t)\}$ is some parameterization of $\Gamma$. Clearly, our definition of $\partial_z|_\Gamma$ is independent of the choice of the parameterization.

With this preliminary, by regarding $D_x^2$ as the holomorphic differential operator $\partial_z^2$ on $\mathbb{C}$, we can restrict the operator $H_V$ to any curve $\Gamma \subset \mathbb{C}$ with the property that

$$\text{Supp } V \subset \Gamma \cap \mathbb{R} .$$

The curve $\Gamma$ inherits a measure from the Lebesgue measure on $\mathbb{C}$. We define $L^2(\Gamma)$ using this measure.

**Theorem 1.8** *Fix* $0 < \theta < \frac{\pi}{2}$. *Assume* Supp $V \subset \{|x| < R\}$. *Let* $\Gamma_\theta$ *be a curve on* $\mathbb{C}$ *satisfying* $\Gamma_\theta \cap \{|z| \leq R\} = [-R, R]$ *and* $\Gamma_\theta \cap \left\{ \begin{array}{c} |z| \geq 2R \\ \pm \mathrm{Re}\, z \geq 0 \end{array} \right\} = \pm e^{i\theta}[2R, \infty)$.

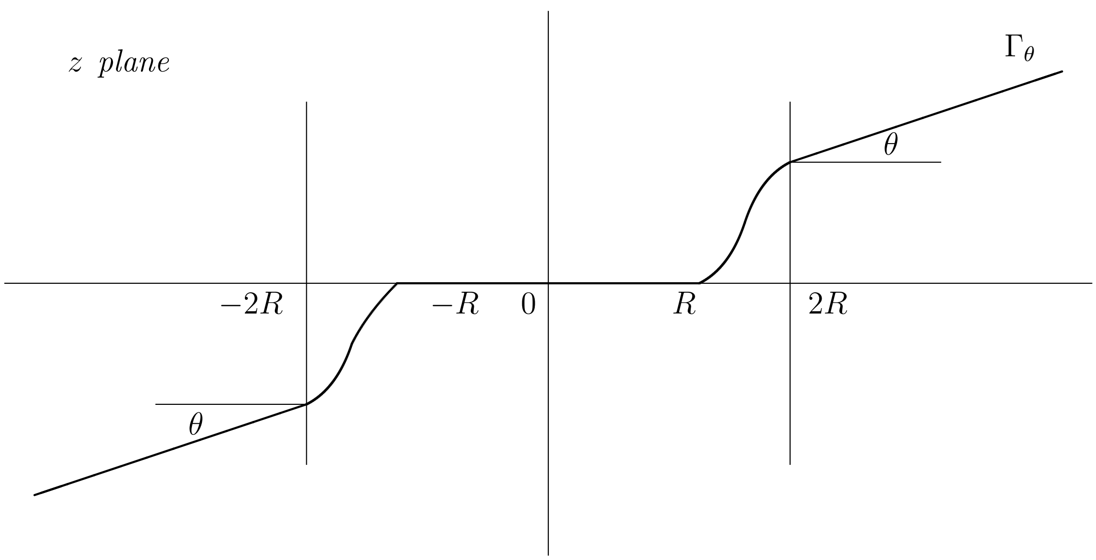


Figure 10

*Put* $H_\theta \stackrel{\text{def}}{=} H_V|_{\Gamma_\theta}$. *Then for* $-\theta < \arg\lambda < 0$ *(recall* $z = \lambda^2$ *is the spectral parameter), the spectrum of* $H_\theta$ *coincides with the resonances of* $H_V$ *in agreement with multiplicity.*

**Proof.** For simplicity, we write $\Gamma_\theta$ as $\Gamma$ below. First we want to show that the spectrum of $H_\theta$ is discrete or, in other words, $(H_\theta - \lambda^2)^{-1}$ is meromorphic for $-\theta < \arg\lambda < \varepsilon$, where $\varepsilon$ is any positive number. As in the proof of Theorem 1.1, this is the same as showing that $(I + V(D_z^2|_\Gamma - \lambda^2)^{-1})^{-1}$ is meromorphic for $-\theta < \arg\lambda < \varepsilon$. By analytic Fredholm theory from the Appendix, it will follow from the compactness of the operator $V(D_z^2|_\Gamma - \lambda^2)^{-1}$ on $L^2(\Gamma)$ for $-\theta < \arg\lambda < \varepsilon$ and the bound

$$\|V(D_z^2|_\Gamma - \lambda^2)^{-1}\rho\|_{L^2\to L^2} \le \frac{C}{|\lambda|} \quad \text{for} \quad \text{Im}\ \lambda > 0 \tag{1.100}$$

for any $\rho \in C_0^\infty(\mathbb{R})$ with $\rho V \equiv V$ which will guarantee the existence of $(I + V(D_z^2|_\Gamma - \lambda^2)^{-1})^{-1}$ for large $\lambda$ with Im $\lambda > 0$ by a Neumann series argument.

Now by (1.9) we have

$$((D_z^2|_\Gamma - \lambda^2)^{-1}u(x) = \frac{2i}{\lambda}\int_\Gamma e^{i\lambda((x-y)^2)^{\frac{1}{2}}}u(y)dy$$

where the branch of square root is chosen to be positive on the positive real axis.
For $|y| >> 0$ on $\Gamma$, we have

$$y = \pm e^{i\theta}r \qquad \text{for} \quad r > 0\ ,$$

thus $((x-y)^2)^{\frac{1}{2}} \sim e^{i\theta}r$ for $|x| < R$ and $r >> 0$. Hence, for $-\theta < \arg\lambda < \varepsilon$, $|x| < R$, we have

$$|e^{i\lambda((x-y)^2)^{\frac{1}{2}}}| \sim e^{-r|\lambda|\sin(\theta+\arg\lambda)}$$

decays exponentially as $r \to \infty$. From this, we see that $V(D_z^2|_\Gamma - \lambda^2)^{-1}$ defines a compact operator on $L^2(\Gamma)$. Next to see (1.100), we simply observe that

$$V((D_z^2|_\Gamma - \lambda^2)^{-1}\rho \equiv V(D_x^2 - \lambda^2)^{-1}\rho \qquad \text{if} \quad \text{supp}\rho \subset \{|x| < R\}$$

and (1.100) then follows from (1.90).

Having now established that the spectrum of $H_\theta$ is discrete for $-\theta < \arg\lambda < \varepsilon$, we have to show that it coincides with the resonance set of $H_V$ there. To make the argument clear, we will assume simple eigenvalue or resonance. The argument for multiplicity follows the same line.

Now, suppose $\lambda_0$ is a simple resonance of $H_V$ which means that $\hat{X}(\lambda)$ has a simple zero at $\lambda_0$ and hence (see (1.50))

$$\phi_-(x,\lambda_0) = \begin{cases} C\, e^{i\lambda_0 x} & x >> 0 \\ e^{-i\lambda_0 x} & x << 0 \end{cases}.$$

For $|x| \geq R$, $\phi_-(x, \lambda_0)$ clearly continues analytically in $x$ to $\mathbb{C}$ and satisfies $(D_z^2 - \lambda_0)\phi_-(z, \lambda_0) = 0$ there. This means that $(H_\theta - \lambda_0^2)(\phi_-|_\Gamma) = 0$. Note also that

$$e^{\pm iz\lambda_0}|_{\Gamma \cap \{|z| \geq R, \pm \mathrm{Re}\, z \geq 0\}} \in L^2(\Gamma \cap \{|z| \geq R, \pm\mathrm{Re}\, z \geq 0\})$$

thus $\phi_-|_\Gamma \in L^2(\Gamma)$ and is an eigenfunction of $H_\theta$ with eigenvalue $\lambda_0^2$. The eigenvalue is simple otherwise the solution of $(H_\theta - \lambda_0^2)^2 u = 0$, $u \in L^2(\Gamma)$ would give a solution to $(H_V - \lambda_0^2)^2 v = 0$ with

$$v = \begin{cases} A\, e^{i\lambda_0 x} & x >> 0 \\ B\, e^{-i\lambda_0 x} & x << 0 \end{cases}.$$

contradicting the simplicity of the resonance (compare with the proof of Proposition 1.9). Similarly, a simple eigenvalue of $H_\theta$ corresponds to a simple resonance of $H_V$.

## 1.5 Trace Formulae

To motivate the trace formulae, we again consider the Dirichlet realization of $H_V$ on a compact interval $[a, b]$, denoting the corresponding self-adjoint operator by $H_V^D$. The spectrum of $H_V^D$ is discrete, $E_N < E_{N-1} < \cdots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty$. Then for $f \in S(\mathbb{R})$, we have

$$\mathrm{tr}\, f(H_V^D) = \sum_{j=0}^{\infty} f(\lambda_j^2) + \sum_{k=1}^{N} f(E_k) \tag{1.101}$$

and

$$\mathrm{tr}\, f(H_V^D) = \int_0^\infty g(\lambda) \frac{dN}{d\lambda}(\lambda) d\lambda + \sum_{k=1}^{N} f(E_k) \tag{1.102}$$

where $N(\lambda) = \#\{\lambda_j^2 : \lambda_j^2 \leq \lambda^2\}$ is the positive eigenvalues counting function and $g(\lambda)$ is the even function defined by $g(\lambda) = f(\lambda^2)$.

In this section we prove the scattering theoretical analogues of (1.101) and (1.102). Our purpose is to present the simple one-dimensional case as a preparation to the results in higher dimensions. In higher dimensions, trace formulae link the "geometry" of the scatterer with the spectral and scattering data. More precisely, trace of $f(H_V)$ can be related to dynamical information. We will see this later.

Although for $H_V^D$, (1.101) and (1.102) are essentially the same once we observe that

$$\frac{dN}{d\lambda}(\lambda) = \sum_{j=0}^{\infty} \delta(\lambda - \lambda_j),$$

their scattering analogues are nevertheless quite different. We start with the analogue of (1.102).

**Theorem 1.9** (Birman Krein Formula) *Suppose* $f \in S(\mathbb{R})$ *and* $g(\lambda) = f(\lambda^2)$. *Then* $f(H_V) - f(H_0)$ *is a trace class operator and*

$$tr(f(H_V) - f(H_0)) = \int_0^\infty g(\lambda)\frac{d\sigma}{d\lambda}(\lambda)d\lambda + \sum_{k=1}^{N} f(E_k) + \frac{\theta}{2}g(0) \qquad (1.103)$$

*where* $\theta = 1$ *if* $\hat{X}(0) \neq 0$ *or 0 if* $\hat{X}(0) = 0$, *also*

$$\sigma(\lambda) = \frac{1}{2\pi i}\log\, det\, S(\lambda) \quad \textit{with} \;\; \sigma(0) = 0 \qquad (1.104)$$

**Proof.** To see the trace class property, we apply the Helffer-Sjöstrand formula in the Appendix and the relevant identity to get

$$f(H_V) - f(H_0) = \frac{-1}{\pi}\int \bar{\partial}_z\tilde{f}(z)(H_V - z)^{-1}V(H_0 - z)^{-1}dz \qquad f \in S(\mathbb{R}) \; . \qquad (1.105)$$

Recall that $\tilde{f}$ is an almost analytic extension fo $f$ and satisfies $|\bar{\partial}\tilde{f}(z)| \leq C_N|\mathrm{Im} z\,|^N\langle z\rangle^{-N}$ for all $N \in \mathbb{N}$.

Since $H_0(H_0 - z)^{-1} = I + z(H_0 - z)^{-1}$, we have

$$\begin{aligned} \|H_0\|_{H^2\to L^2}\|(H_0 - z)^{-1}\|_{L^2\to H^2} &= \|H_0((H_0 - z)^{-1}\|_{L^2\to L^2} \\ &= \|I + z(H_0 - z)^{-1}\|_{L^2\to L^2} \geq 1 + \frac{|z|}{|\mathrm{Im}\, z|} \end{aligned}$$

which implies

$$V(H_0 - z)^{-1} = O\left(\frac{|z|}{|\mathrm{Im}\, z|}\right) : L^2 \to H^2_{\mathrm{comp}}([-R, R]) \qquad (1.106)$$

where $\mathrm{Supp}\; V \subset [-R, R]$.

The results in the Appendix implies that $V(H_0 - z)^{-1}$ is of trace class and

$$\|(H_V - z)^{-1}V(H_0 - z)^{-1}\|_{\mathrm{tr}} \leq \frac{C\langle z\rangle}{|\mathrm{Im}\, z|^2} \qquad (1.107)$$

Combining (1.105), (1.106), (1.107), we have

$$f(H_V) - f(H_0) \in L_1(L^2(\mathbb{R}), L^2(\mathbb{R})) \; .$$

Now we can prove (1.103). For simplicity, we assume that $H_V$ has no negative eigenvalue (as their contribution is quite clear). Thus, by Theorem 1.2, we can write

$$f(H_V)(x, x) = \frac{1}{2\pi}\int_0^\infty (e_+(x, \lambda)e_+(x, -\lambda) + e_-(x, \lambda)e_-(x, -\lambda))g(\lambda)d\lambda \; , \qquad (1.108)$$

and we also have

$$f(H_0)(x,x) = \frac{1}{2\pi}\int_0^\infty 2g(\lambda)d\lambda\ . \tag{1.109}$$

Using Lidskii's Theorem in the Appendix and (1.108),(1.109), we get

$$\begin{aligned}
\mathrm{tr}(f(H_V) - f(H_0)) &= \lim_{r\to\infty}\int_{-r}^{r}(f(H_V)(x,x) - f(H_0)(x,x))dx \\
&= \frac{1}{2\pi}\lim_{r\to\infty}\int_0^\infty\int_{-r}^{r}[e_+(x,\lambda)e_+(x,-\lambda)+(e_-(x,\lambda)e_-(x,-\lambda) - 2]dx\, g(\lambda)d\lambda \quad (1.110)\\
&= \frac{1}{4\pi}\lim_{r\to\infty}\int_{-\infty}^\infty\int_{-r}^{r}[e_+(x,\lambda)e_+(x,-\lambda)+(e_-(x,\lambda)e_-(x,-\lambda) - 2]dx\, g(\lambda)d\lambda
\end{aligned}$$

To eliminate the integration in $x$, we apply the following "reduction to boundary" trick. The resulting formula (1.114) is known as the Maass-Selberg Formula. We have

$$(H_V - \lambda^2)e_\pm(x,\lambda) = 0\ . \tag{1.111}$$

Differentiating with respect to $\lambda$, we get

$$(H_V - \lambda^2)\partial_\lambda e_\pm(x,\lambda) = 2\lambda e_\pm(x,\lambda)\ . \tag{1.112}$$

Hence, for $\lambda \neq 0$,

$$\begin{aligned}
&e_\pm(x,\lambda)e_\pm(x,-\lambda) \\
&\quad= \frac{(H_V - \lambda^2)}{2\lambda}(\partial_\lambda e_\pm(x,\lambda))e_\pm(x,-\lambda) - \frac{1}{2\lambda}(\partial_\lambda e_\pm(x,\lambda))(H_V - \lambda^2)e_\pm(x,-\lambda) \\
&\quad= \frac{1}{2\lambda}(D_x^2(\partial_\lambda e_\pm(x,\lambda))e_\pm(x,-\lambda) - (\partial_\lambda e_\pm(x,\lambda))D_x^2 e_\pm(x,-\lambda)) \quad (1.113)
\end{aligned}$$

where we have used (1.111) and (1.112) in the first equality. Thus,

$$\begin{aligned}
&\int_{-r}^{r} e_\pm(x,\lambda)e_\pm(x,-\lambda)dx \\
&\quad= \frac{1}{2\lambda}\int_{-r}^{r}[-\partial_x^2(\partial_\lambda e_\pm(x,\lambda))e_\pm(x,-\lambda) + \partial_\lambda e_\pm(x,\lambda)\partial_x^2 e_\pm(x,-\lambda)]dx \\
&\quad= \frac{1}{2\lambda}\int_{-r}^{r}\partial_x(\partial_\lambda e_\pm(x,\lambda)\partial_x e_\pm(x,-\lambda) - \partial_x\partial_\lambda e_\pm(x,\lambda)e_\pm(x,-\lambda))dx \\
&\quad= \frac{1}{2\lambda}\ [(\partial_\lambda e_\pm(x,\lambda))(\partial_x e_\pm(x,-\lambda)) - (\partial_x\partial_\lambda e_\pm(x,\lambda))e_\pm(x,-\lambda)]_{-r}^{r}\ . \quad (1.114)
\end{aligned}$$

Put this into (1.110), we get

$$\operatorname{tr}(f(H_V) - f(H_0)) = \frac{1}{8\pi} \lim_{\varepsilon \to 0, r \to \infty} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \tag{1.115}$$

$$\left( \sum_{\pm} [(\partial_\lambda e_\pm(x,\lambda))(\partial_x e_\pm(x,-\lambda)) - (\partial_x \partial_\lambda e_\pm(x,\lambda)) e_\pm(x,-\lambda)]_{-r}^{r} - 8r\lambda \right) \frac{g(\lambda)}{\lambda} d\lambda$$

Now, as we know the behaviors of $e_\pm(x,\lambda)$ for $|x| >> 0$ precisely, the remainder of the proof amounts to a direct, but quite tedious computation.

First, we record the formulae of $e_\pm(x,\lambda)$ and their first derivatives for $|x| >> 0$. In the following formulae, as a rule, the upper row gives the behavior for $x >> 0$ and the lower row for $x << 0$,

$$\begin{aligned}
e_+(x,\lambda) &= \begin{cases} T(\lambda) e^{i\lambda x} \\ e^{i\lambda x} + R_+(\lambda) e^{-i\lambda x} \end{cases} \\
e_-(x,\lambda) &= \begin{cases} e^{-i\lambda x} + R_-(\lambda) e^{i\lambda x} \\ T(\lambda) e^{-i\lambda x} \end{cases} \\
\partial_x e_+(x,\lambda) &= \begin{cases} i\lambda T(\lambda) e^{i\lambda x} \\ i\lambda e^{i\lambda x} - i\lambda R_+(\lambda) e^{-i\lambda x} \end{cases} \\
\partial_x e_-(x,\lambda) &= \begin{cases} -i\lambda e^{-i\lambda x} + i\lambda R_-(\lambda) e^{i\lambda x} \\ -i\lambda T(\lambda) e^{-i\lambda x} \end{cases} \\
\partial_\lambda e_+(x,\lambda) &= \begin{cases} (T'(\lambda) + ixT(\lambda)) e^{i\lambda x} \\ ixe^{i\lambda x} + (R'_+(\lambda) - ixR_+(\lambda)) e^{-i\lambda x} \end{cases} \\
\partial_\lambda e_-(x,\lambda) &= \begin{cases} -ixe^{-i\lambda x} + (R'_-(\lambda) + ixR_-(\lambda)) e^{i\lambda x} \\ (T'(\lambda) - ixT(\lambda)) e^{-i\lambda x} \end{cases}
\end{aligned} \tag{1.116}$$

Now, by using (1.116), we compute

$$
\begin{aligned}
&\sum_{\pm}(\partial_\lambda e_\pm(x,\lambda))(\partial_x e_\pm(x,-\lambda))|_{-r}^{r} \\
&= (T'(\lambda)+irT(\lambda))e^{i\lambda r}(-i\lambda T(-\lambda)e^{-i\lambda r}) \\
&\quad -(-ire^{-i\lambda r}+(R_+'(\lambda)+irR_+(\lambda))e^{i\lambda r})(-i\lambda e^{i\lambda r}+i\lambda R_+(-\lambda)e^{-i\lambda r}) \\
&\quad +(-ire^{-i\lambda r}+(R_-'(\lambda)+irR_-(\lambda))e^{i\lambda r})(i\lambda e^{i\lambda r}-i\lambda R_-(-\lambda)e^{-i\lambda r}) \\
&\quad (T'(\lambda)+irT(\lambda))e^{i\lambda r}(i\lambda T(-\lambda)e^{-i\lambda r}) \\
&= -i\lambda(T'(\lambda)T(-\lambda)+irT(\lambda)T(-\lambda)) - [-r\lambda + i\lambda R_+(-\lambda)(R_+'(\lambda)+irR_+(\lambda)) \\
&\quad +r\lambda R_+(-\lambda)e^{-2i\lambda r} - i\lambda(R_+'(\lambda)+irR_+(\lambda))e^{2i\lambda r}] \\
&\quad -i\lambda(T'(\lambda)T(-\lambda)+irT(\lambda)T(-\lambda)) - [-r\lambda + i\lambda R_-(-\lambda)(R_-'(\lambda)+irR_-(\lambda)) \\
&\quad r\lambda R_-(-\lambda)e^{-2i\lambda r} - i\lambda(R_-'(\lambda)+irR_-(\lambda))e^{2i\lambda r}] \\
&= -i\lambda(2T'(\lambda)T(-\lambda)+R_+'(\lambda)R_+(-\lambda)+R_-'(\lambda)R_-(-\lambda)) + 4r\lambda \\
&+ \text{ terms of the form } \lambda h(r,\lambda)e^{\pm 2i\lambda r} \text{ with smooth and tempered function } h
\end{aligned}
$$

Note that

$$
\lim_{\varepsilon\to 0, r\to\infty}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \lambda h(r,\lambda)e^{\pm 2i\lambda r}\frac{g(\lambda)}{\lambda}\,d\lambda = 0
$$

as $g\in S(\mathbb{R})$. Thus,

$$
\begin{aligned}
&\frac{1}{8\pi}\lim_{\varepsilon\to 0, r\to\infty}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\left(\sum_{\pm}(\partial_\lambda e_\pm(x,\lambda))(\partial_x e_\pm(x,-\lambda))|_{-r}^{r} - 4r\lambda\right)\frac{g(\lambda)}{\lambda}\,d\lambda \\
&\quad = \frac{1}{8\pi i}\int_{\mathbb{R}}(2T'(\lambda)T(-\lambda)+R_+'(\lambda)R_+(-\lambda)+R_-'(\lambda)R_-(-\lambda))g(\lambda)d\lambda \qquad (1.117) \\
&\quad = \frac{1}{4}\int_{\mathbb{R}}\frac{d\sigma}{d\lambda}\,g(\lambda)d\lambda
\end{aligned}
$$

Here, we have used the expression of $S(\lambda)$ given in (1.56) to compute $\frac{d\sigma}{d\lambda}$. Next, we look at

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\left(\left[\sum_{\pm}-(\partial_x\partial_\lambda e_\pm(x,\lambda))e_\pm(x,-\lambda)\right]_{-r}^{r}-4r\lambda\right)\frac{g(\lambda)}{\lambda}\,d\lambda$$

$$=\lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\left[\sum_{\pm}-\partial_\lambda((\partial_x e_\pm(x,\lambda))e_\pm(x,-\lambda)\right]_{-r}^{r}$$

$$-\left(\sum_{\pm}(\partial_x e_\pm(x,\lambda))((\partial_\lambda e_\pm)(x,-\lambda)|_{-r}^{r}-4r\lambda\right)\frac{g(\lambda)}{\lambda}\,d\lambda$$

$$=: (I)+(II)$$

where the splitting into two sums corresponds to the two summation signs. By a change of variables, $\lim_{r\to\infty}\frac{(II)}{8\pi}$ is exactly the same as the term computed in (1.117). Thus, so far we obtain

$$\mathrm{tr}(f(H_V)-f(H_0))=\tfrac{1}{2}\int_{\mathbb{R}}g(\lambda)\,\frac{d\sigma}{d\lambda}\,d\lambda+\frac{1}{8\pi}\lim_{r\to\infty}(I)\ . \tag{1.118}$$

Finally, we have to compute

$$\frac{1}{8\pi}\lim_{r\to\infty}(I)=\frac{1}{8\pi}\lim_{\varepsilon\to 0,r\to\infty}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\sum_{\pm}-\partial_\lambda((\partial_x e_\pm(x,\lambda))e_\pm(x,-\lambda))|_{-r}^{r}\frac{g(\lambda)}{\lambda}\,d\lambda$$

Using (1.116) we get

$$\begin{aligned}&\sum_{\pm}-\partial_\lambda(\partial_x e_\pm(x,\lambda))e_\pm(x,-\lambda))|_{-r}^{r}\\ =\quad&-\partial_\lambda\{i\lambda[(R_+(\lambda)+R_-(\lambda))e^{2i\lambda r}-(R_+(-\lambda)+R_-(-\lambda))e^{-2i\lambda r}]\}\end{aligned}$$

As before, if we integrate the term

$$-i\lambda\partial_\lambda\{(R_+(\lambda)+R_-(\lambda))e^{2i\lambda r}-(R_+(-\lambda)+R_-(-\lambda))e^{-2i\lambda r}\}\,\frac{g(\lambda)}{\lambda}$$

in $\lambda$ and let $r\to\infty$, we get zero. Hence

$$\begin{aligned}&\frac{1}{8\pi}\lim_{\varepsilon\to 0,r\to\infty}(I)\\ =\quad&\frac{1}{8\pi}\lim_{\varepsilon\to 0,r\to\infty}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}\sum_{\pm}-\partial_\lambda((\partial_x e_\pm(x,\lambda))e_\pm(x,-\lambda))|_{-r}^{r}\,\frac{g(\lambda)}{\lambda}\,d\lambda\\ =\quad&\frac{1}{8\pi i}\lim_{\varepsilon\to 0,r\to\infty}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}[(R_+(\lambda)+R_-(\lambda))e^{2i\lambda r}-(R_+(-\lambda)+R_-(\lambda))e^{-2i\lambda r}]\,\frac{g(\lambda)}{\lambda}\,d\lambda\\ =\quad&\frac{1}{4\pi i}\lim_{\varepsilon\to 0,r\to\infty}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}[(R_+(\lambda)+R_-(\lambda))e^{2i\lambda r}\,\frac{g(\lambda)}{\lambda}\,d\lambda\end{aligned} \tag{1.119}$$

In computing (1.119), we may assume $g$ is entire by approximating $f$ by Schwartz functions with compactly supported Fourier transform side, and deform the contour of integration as shown in the following Figure 11.

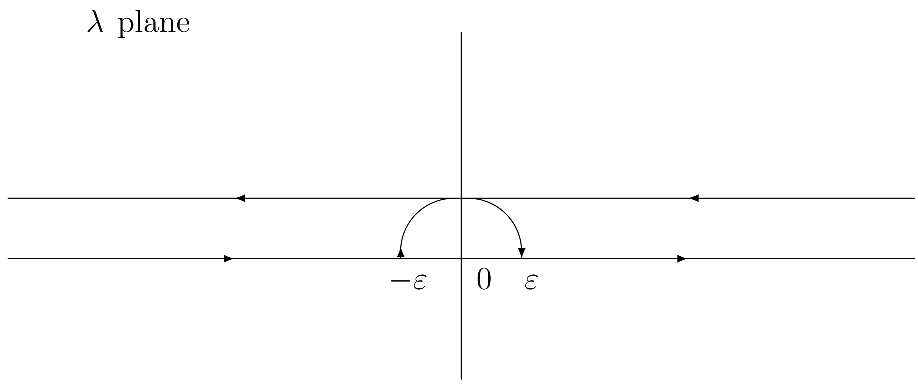


Figure 11

This is admissible as $R_+(\lambda)$ and $R_-(\lambda)$ are holomorphic in the upper half plane and $e^{2i\lambda r}$ decays exponentially when $r > 0$ there. Hence

$$\begin{aligned}\frac{1}{8\pi}\lim_{\varepsilon\to 0, r\to\infty}(I) &= -\tfrac{1}{4}\left(\mathrm{Res}_0((R_+(\lambda)+R_-(\lambda))e^{2i\lambda r}\,\frac{g(\lambda)}{\lambda}\right) \\ &= \begin{cases} 0 & \text{if } \hat{X}(0)=0 \\ -\tfrac{1}{4}\left(\frac{2\hat{Y}(0)}{\hat{X}(0)}\right)g(0) & \text{if } \hat{X}(0)\neq 0 \end{cases}\end{aligned} \tag{1.120}$$

as $R_\pm(\lambda) = \frac{\hat{Y}(\mp\lambda)}{\hat{X}(\lambda)}$.

Finally we obtain (1.103) by putting (1.120) into (1.118) and observe that $\frac{d\sigma}{d\lambda}$ is an even function and $\hat{Y}(0) = -\hat{X}(0)$ by (1.49) and (1.51).

**Remark.** $\sigma(\lambda)$ defined in (1.104) is called the scattering phase which is a natural analogue of the eigenvalue counting function $N(\lambda)$.

We now give the analogue of (1.101) in terms of resonances.

**Theorem 1.10** (Poisson Formula) *Let $f(\lambda)$ be a function such that if $g(\lambda) = f(\lambda^2)$, we have $\hat{g}(t) \in t^3C_0^\infty([0,\infty))$, then*

$$\mathrm{tr}(f(H_V) - f(H_0)) = \tfrac{1}{2}\sum_{\lambda\in\mathbb{C}} m_R(\lambda)g(\lambda) + g(0)$$

*where* $m_R(0) = \begin{cases} 1 & \textit{if} \ \hat{X}(0) = 0 \\ 0 & \textit{if} \ \hat{X}(0) \neq 0 \end{cases}$

**Proof.** By using Theorem 1.9, we need to show that

$$\int_{-\infty}^{\infty} g(\lambda) \frac{d\sigma}{d\lambda}(\lambda) d\lambda = \sum_{\lambda \in \mathbb{C}} m_R(\lambda) g(\lambda) - 2 \sum_{k=1}^{N} f(E_k) - m_R(0) g(0)$$

where we have used the evenness of $g$ and $\frac{d\sigma}{d\lambda}$.

Recall that

$$\det S(\lambda) = \frac{-\hat{X}(-\lambda)}{\hat{X}(\lambda)} .$$

Then by using Theorem 1.7 and Hadamard factorization, we have

$$\hat{X}(\lambda) = e^{a_0 + a_1 \lambda} P(\lambda) \quad \text{where} \quad P(\lambda) = \prod_{\lambda_j\text{'s are resonances}} \left(1 - \frac{\lambda}{\lambda_j}\right) e^{\frac{\lambda}{\lambda_j}}$$

Let $G$ be a function such that $G' = g$. Note that such $G$ exists and is unique as $\hat{g}(0) = 0$. Then, we have

$$\begin{aligned} \int_{-\infty}^{\infty} g(\lambda) \frac{d\sigma}{d\lambda}(\lambda) d\lambda &= -\int_{-\infty}^{\infty} G(\lambda) \frac{d^2\sigma}{d\lambda^2} d\lambda \\ &= -\int_{-\infty}^{\infty} G(\lambda) \left( \sum_j \left( \frac{1}{(\lambda - \lambda_j)^2} - \frac{1}{(\lambda + \lambda_j)^2} \right) \right) d\lambda \end{aligned}$$

Since $\hat{G} \in t^2 C_0^\infty([0,\infty))$, $|G(\lambda)| \leq \frac{C}{|\lambda|^2}$ for $\operatorname{Im} \lambda \leq 0$. Again, by using the estimate on the number of resonances in Theorem 1.7, we can deform the contour of integration along the lower half plane. We get

$$\begin{aligned} &\int_{-\infty}^{\infty} g(\lambda) \frac{d\sigma}{d\lambda} d\lambda \\ &= \sum_{\lambda \in \mathbb{C}_- \setminus 0} m_R(\lambda) g(\lambda) - \sum_{\lambda \in \mathbb{C}_+} m_R(\lambda) g(\lambda) \\ &= \sum_{\lambda \in \mathbb{C}} m_R(\lambda) g(\lambda) - 2 \sum_{k=1}^{N} f(E_k) - m_R(0) g(0) \end{aligned}$$

which completes the proof.