SEMINAR 9

Some serious maths today ...

$$\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \overset{F}{\rightsquigarrow} \vec{\hat{f}} = \begin{pmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_N \end{pmatrix}$$

F DISCRETE FOURIER TRANSFORM

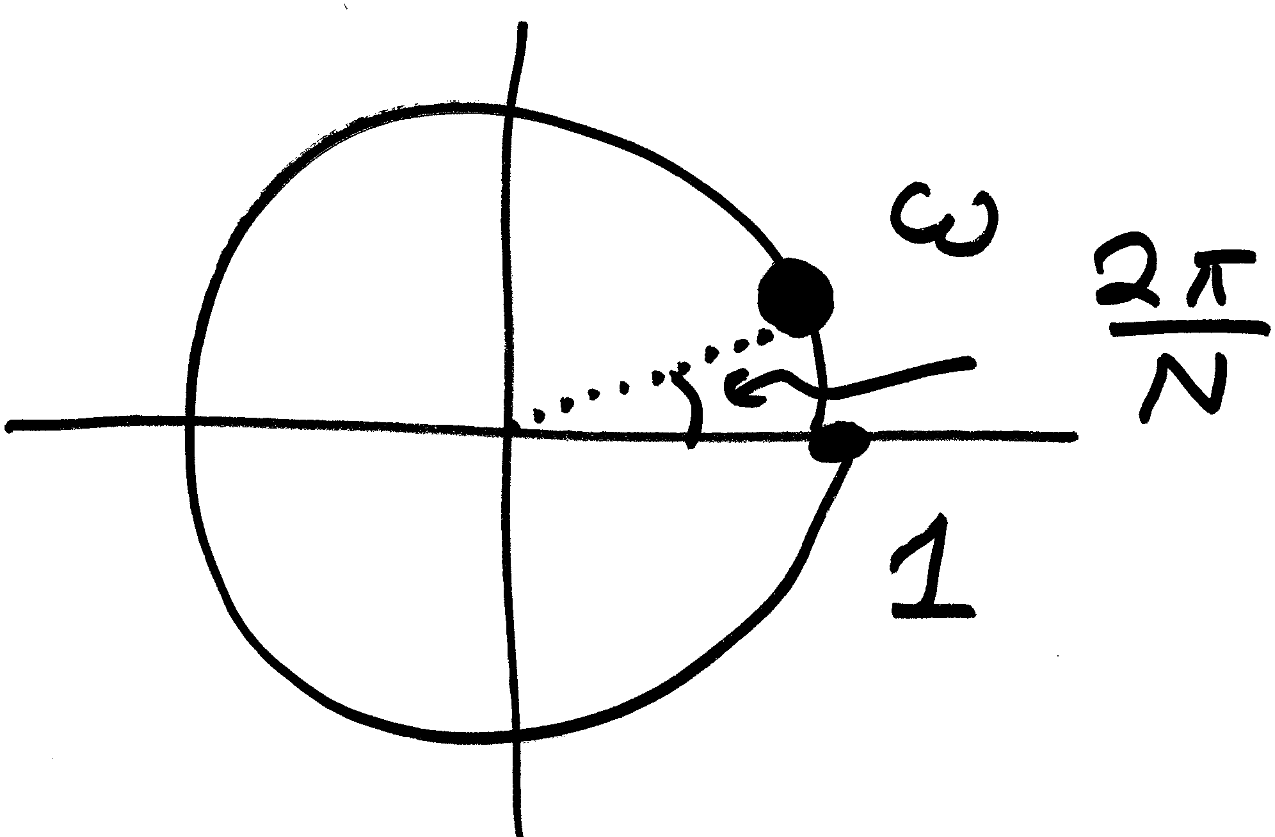
POSITION $\rightsquigarrow$ MOMENTUM

PRECISE MATHEMATICAL FORMULATION

(finally...)

$$\omega = e^{\frac{2\pi i}{N}}$$

N^{th} root of unity



$$\boxed{\hat{f}_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \omega^{-jk} f_j}$$

__THEOREM__ [INVERSION FORMULA]

$$f_j = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \omega^{jk} \hat{f}_k$$

(proof later)

COROLLARY

$$\max |f_j| \le \frac{1}{\sqrt{N}} \sum_k |\hat{f}_k|$$

$$\max |\hat{f}_k| \le \frac{1}{\sqrt{N}} \sum_j |f_j|$$

PROOF

$$|f_j| \le \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |\omega^{jk}| \, |\hat{f}_k|$$

BUT $|\omega^{jk}| = 1$

$$\operatorname{Spt} \vec{f} = \{ j : f_j \neq 0 \}$$

$$\operatorname{Spt} \vec{\hat{f}} = \{ k : \hat{f}_k \neq 0 \}$$

$|A| = \#$ of elements of A

DISCRETE UNCERTAINTY PRINCIPLE: $\vec{f} \neq 0$

$$|\operatorname{Spt} \vec{f}| \cdot |\operatorname{Spt} \vec{\hat{f}}| \geq N$$

---

i.e. $\operatorname{Spt} \vec{f}$ & $\operatorname{Spt} \vec{\hat{f}}$ CANNOT BE SIMULTANEOUSLY SPARSE

PROOF

$$\max |f_j| \le \frac{1}{\sqrt{N}} \sum_k |\hat{f}_k|$$

$$\le \frac{1}{\sqrt{N}} |\operatorname{spt} \vec{\hat{f}}| \max |\hat{f}_k|$$

$$\le \frac{1}{\sqrt{N}} |\operatorname{spt} \vec{\hat{f}}| \frac{1}{\sqrt{N}} \sum_j |f_j|$$

$$\le \frac{1}{N} |\operatorname{spt} \vec{\hat{f}}| \, |\operatorname{spt} \vec{f}| \cdot \max |f_j|$$

CANCEL $\max |f_j| \neq 0$
(as $\vec{f} \neq 0$).

CRUCIAL STEP:

$$\sum_j |a_j|$$

$$\leq |\operatorname{spt} \vec{a}| \max |a_j|$$

only $a_j \neq 0$ contribute

each is $\leq \max |a_j|$