Fractal Weyl laws for resonances of open quantum maps

SCATT05 Dresden

Stéphane Nonnenmacher (Saclay)
and
Maciej Zworski (Berkeley)

March 11, 2005

A simple model of classical chaotic dynamics

Open Baker Relation

A Baker relation (as opposed to a map) is defined on a phase space torus, $0 < q < 1$, $0 < p < 1$:

$$0 < q < 1/3 \implies B(q,p) = \left(3q, \frac{p}{3}\right)$$

$$2/3 < q < 1 \implies B(q,p) = \left(3q-2, \frac{p+2}{3}\right)$$

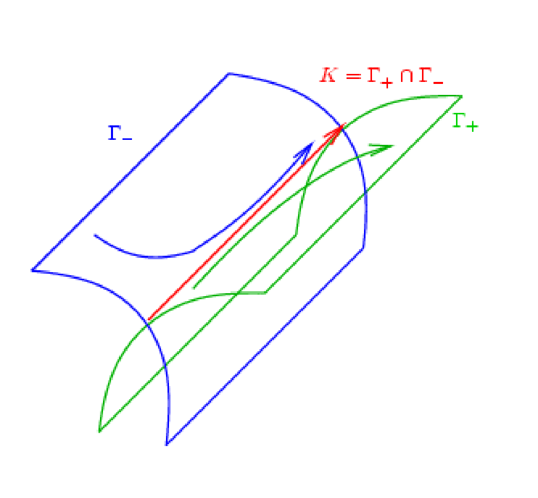
All other points are sent to infinity

Or, we can say that they are not in relation with any other points, $(p,q) \sim B(p,q)$.

We have the outgoing (unstable +) and incoming (stable -) tails defined in the usual fashion:

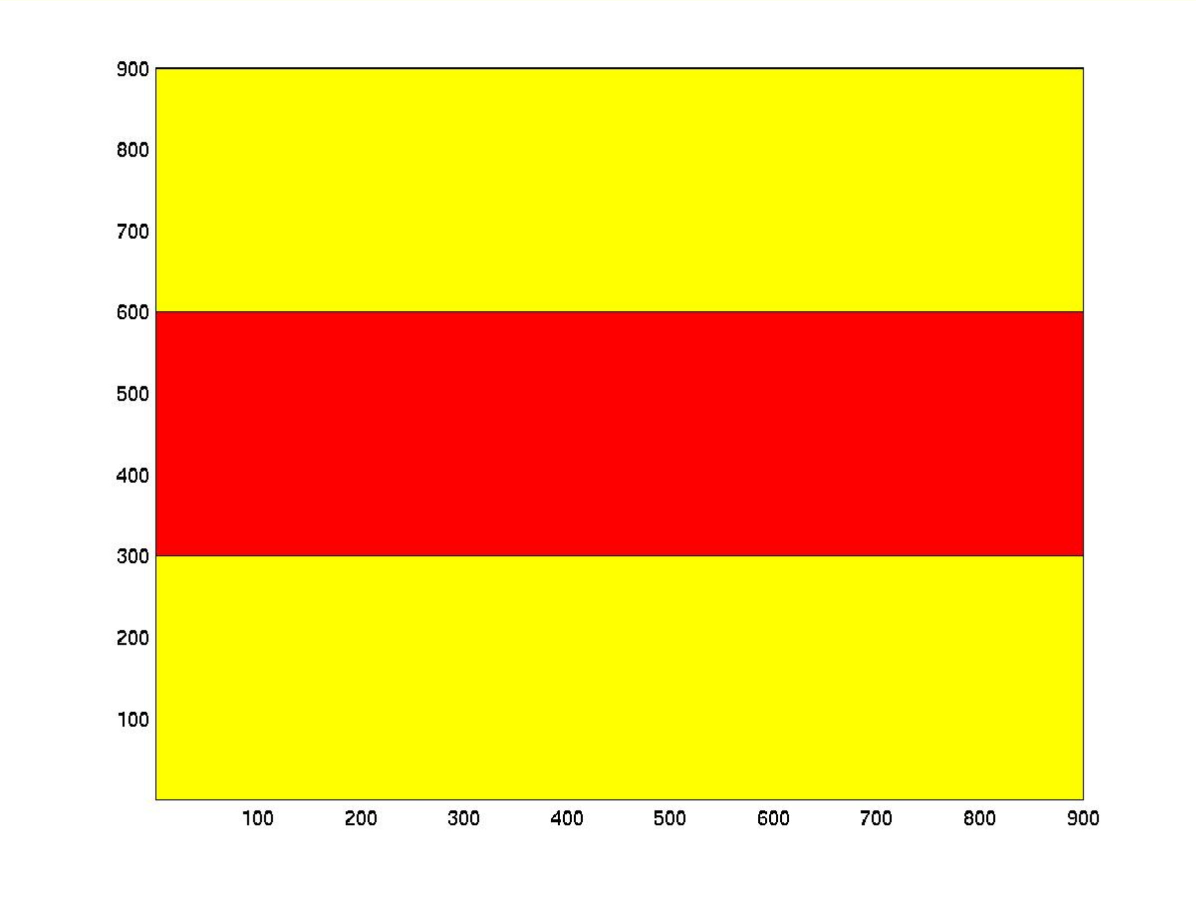
$$(q,p) \in \Gamma_- \iff (p_1, q_1) = B(q,p)\,, \quad (p_{j+1}, q_{j+1}) = B(p_j, q_j)\,,$$

$$(q,p) \in \Gamma_+ \iff (q,p) = B(p_1, q_1)\,, \quad (p_j, q_j) = B(p_{j+1}, q_{j+1})\,,$$



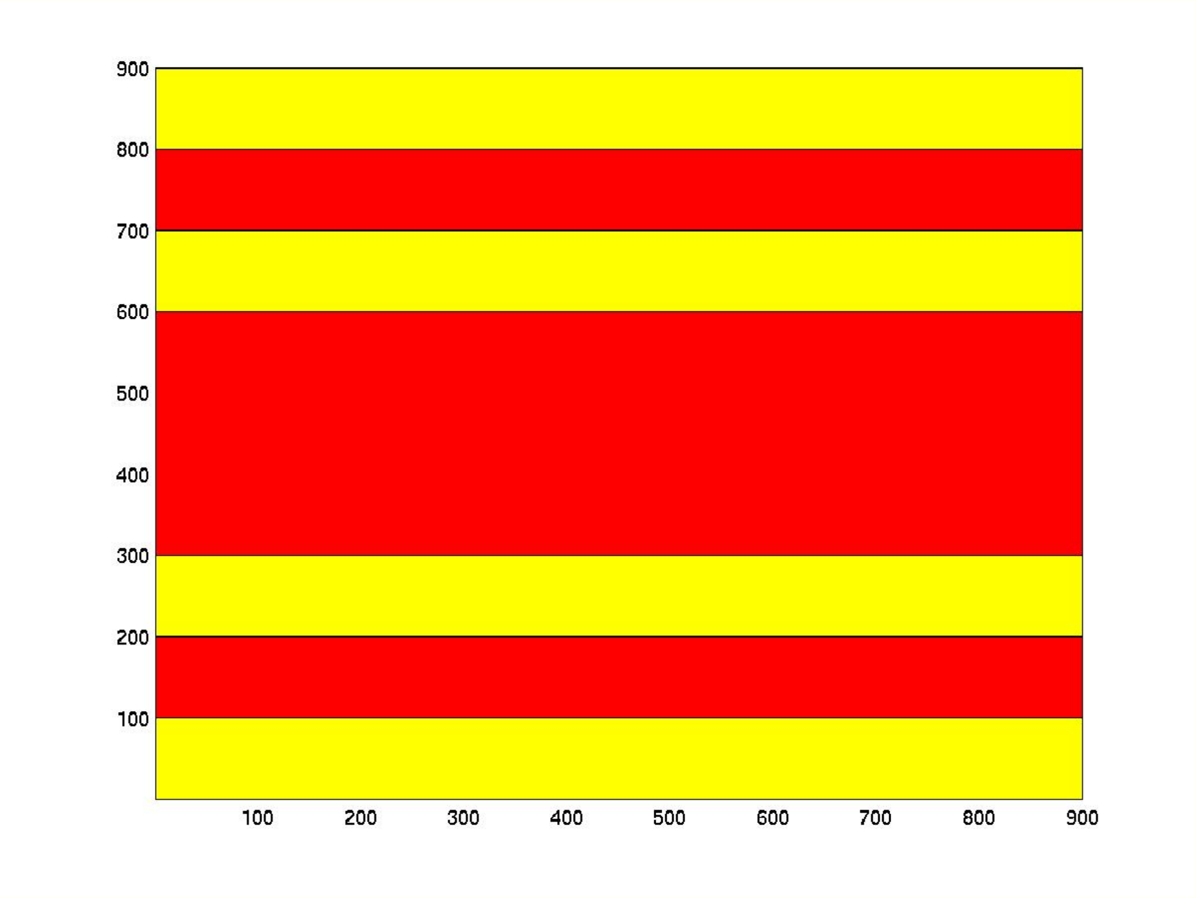
Here are $\Gamma_\pm$ and the trapped set $K = \Gamma_+ \cap \Gamma_-$ in the case of a flow.

Here is how they look like:



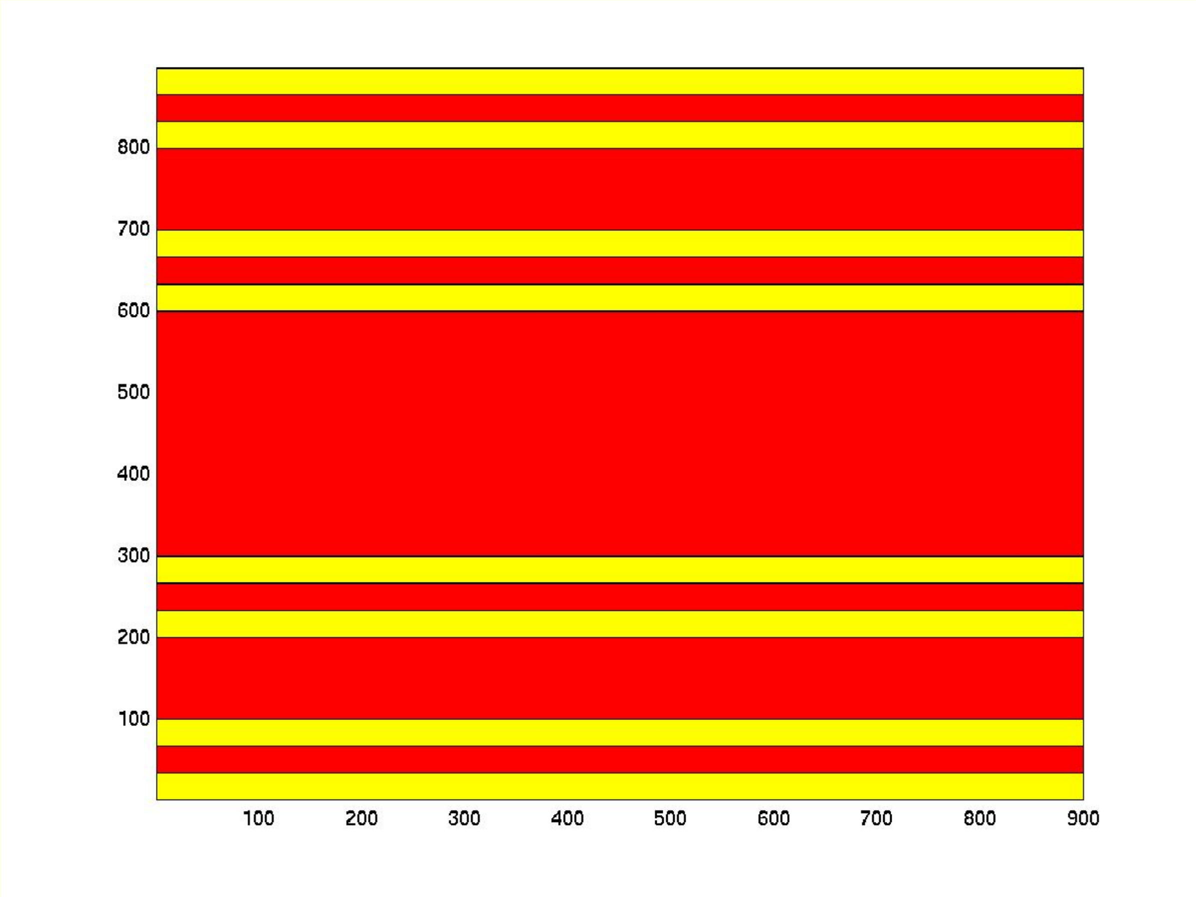
$$\Gamma_+ = \bigcap_{N\geq 0} B^N(\mathbf{T}^2) \subset B(\mathbf{T}^2)$$

Here is how they look like:



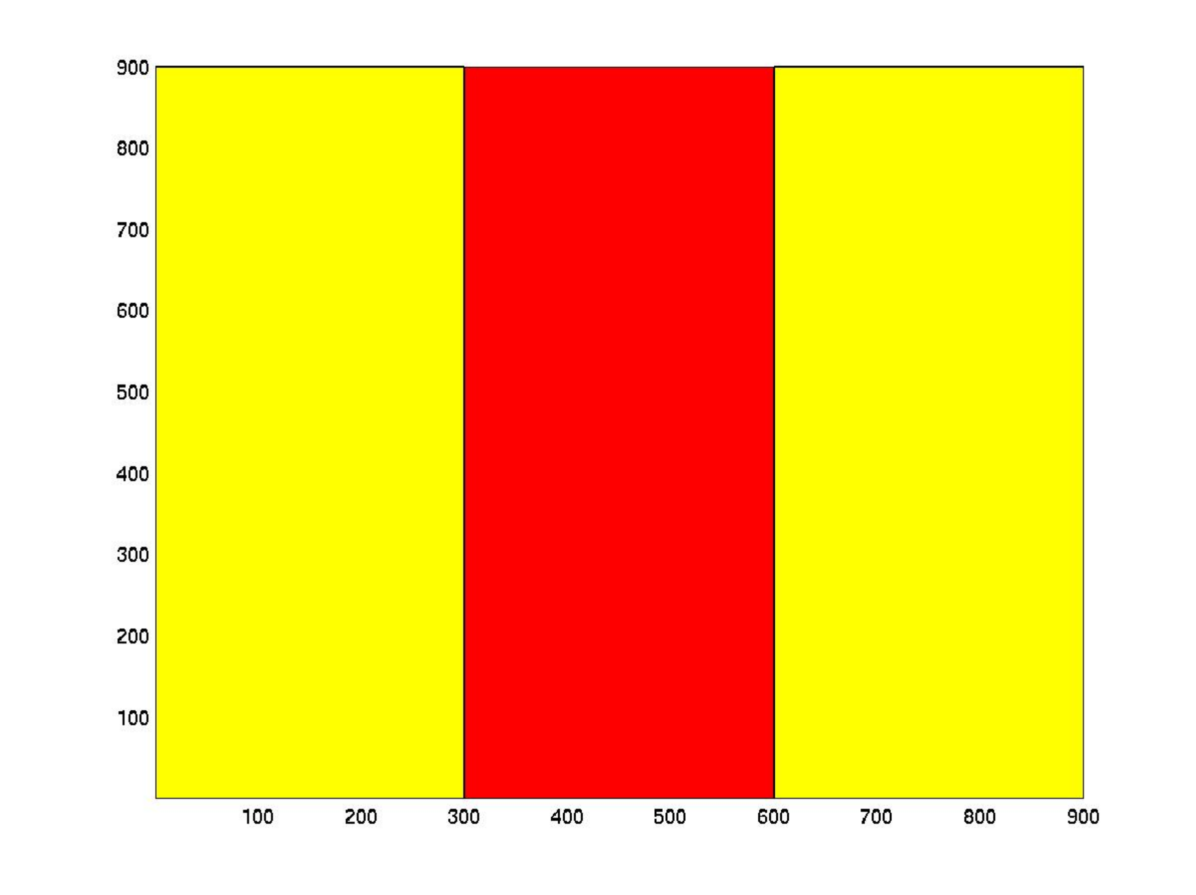
$$\Gamma_+ = \bigcap_{N \geq 0} B^N(\mathbf{T}^2) \subset \bigcap_{N=0,1} (B^N)(\mathbf{T}^2)$$

Here is how they look like:



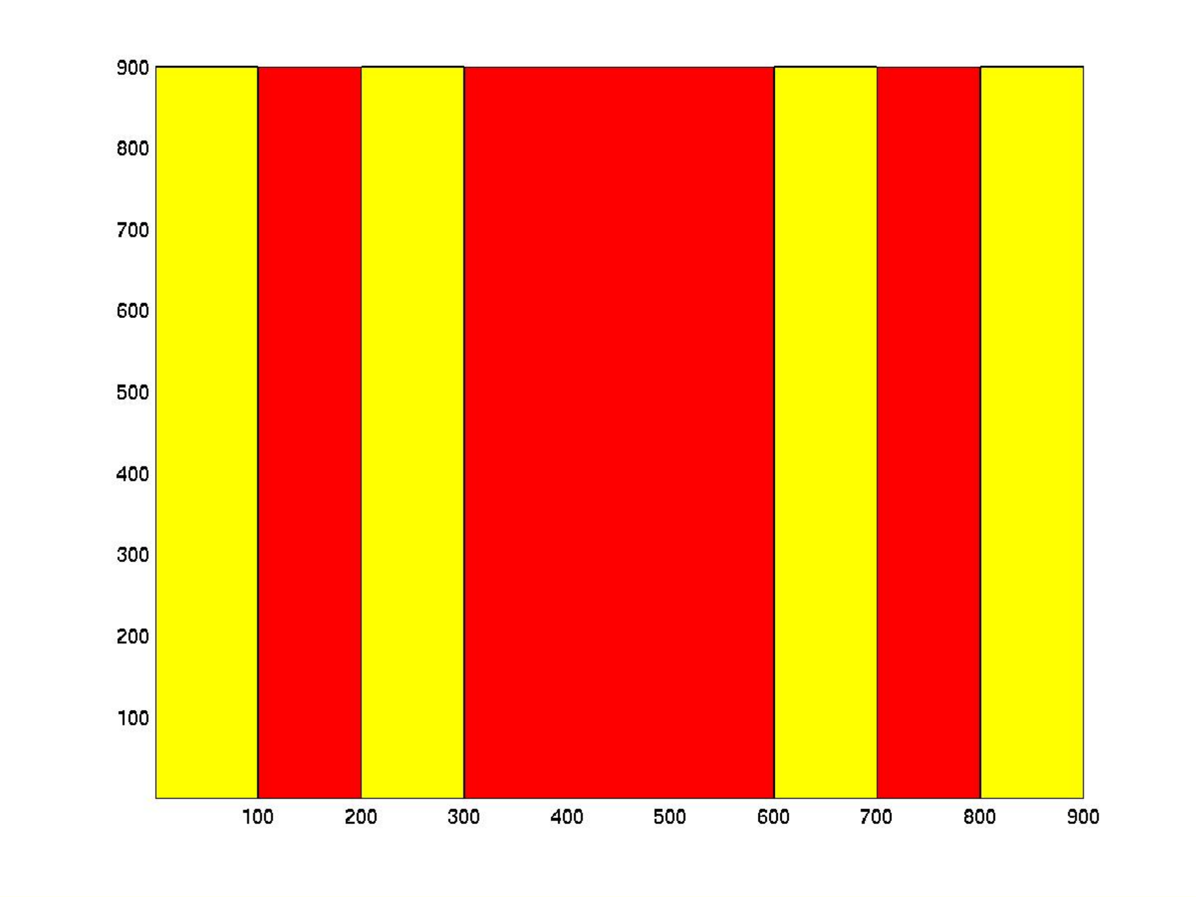
$$\Gamma_+ = \bigcap_{N \geq 0} B^N(\mathbf{T}^2) \subset \bigcap_{N=0,1,3} (B^N)(\mathbf{T}^2)$$

Now for the incoming (stable) tail, Γ_-:



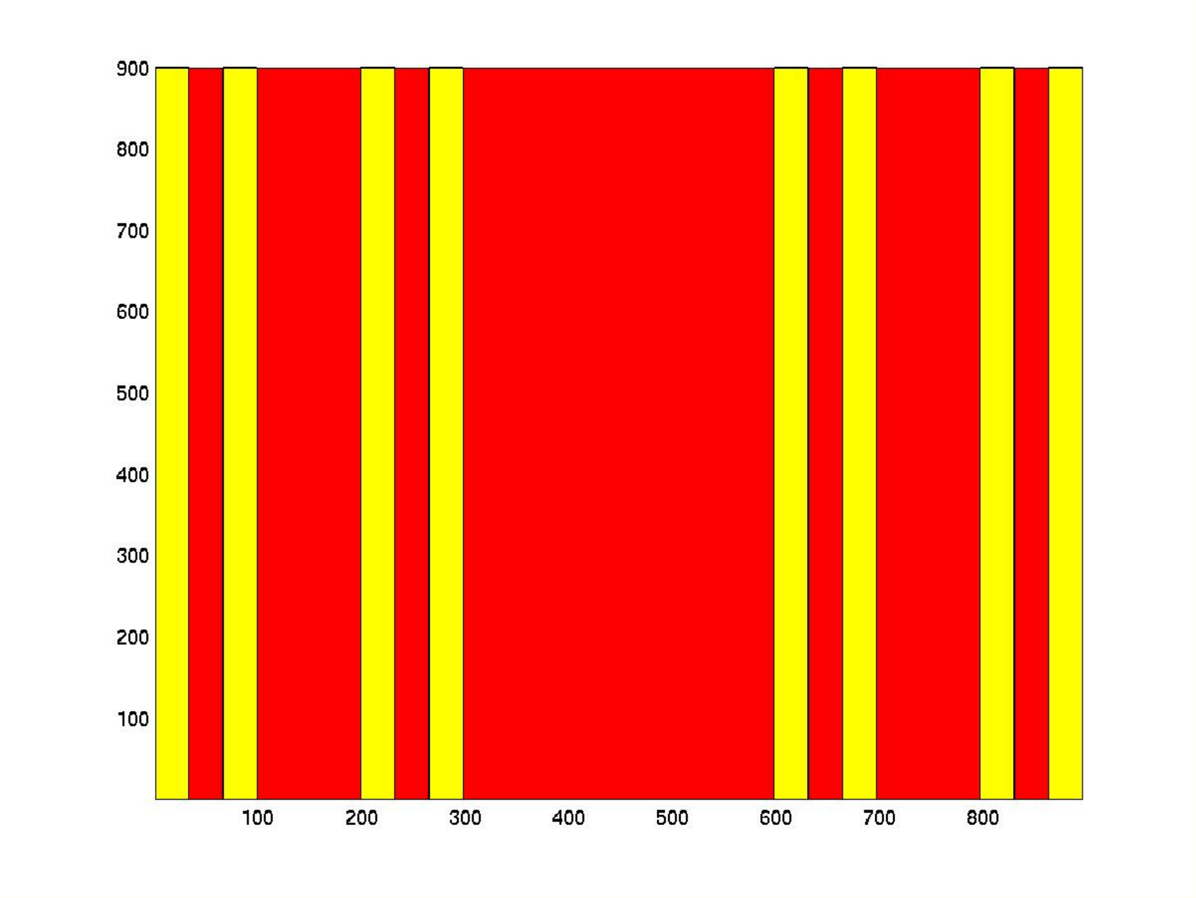
$$\Gamma_- = \bigcap_{N \geq 0} B^{-N}(\mathbf{T}^2) \subset B^{-1}(\mathbf{T}^2)$$

Now for the incoming (stable) tail, Γ_-:



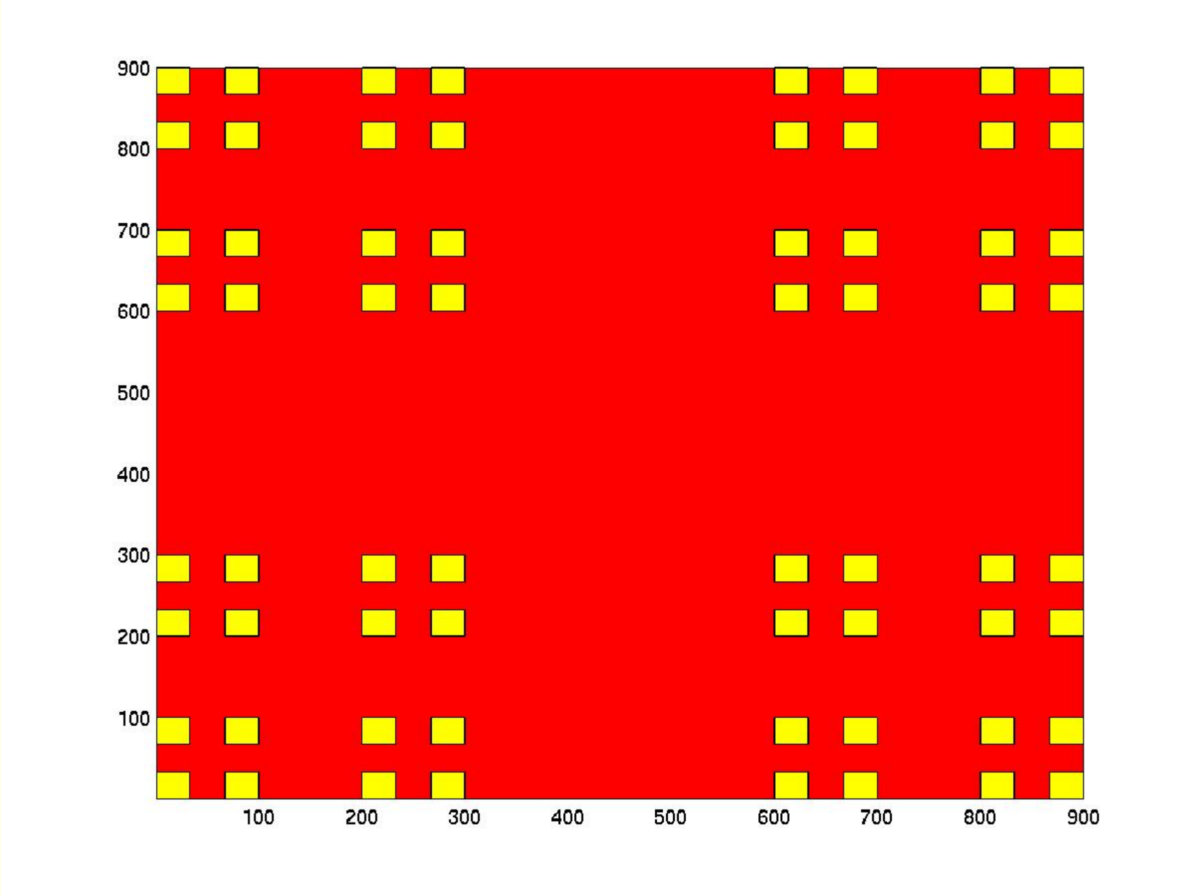
$$\Gamma_- = \bigcap_{N \geq 0} B^{-N}(\mathbf{T}^2) \subset \bigcap_{N=0,1} B^{-N}(\mathbf{T}^2)$$

Now for the incoming (stable) tail, Γ_-:



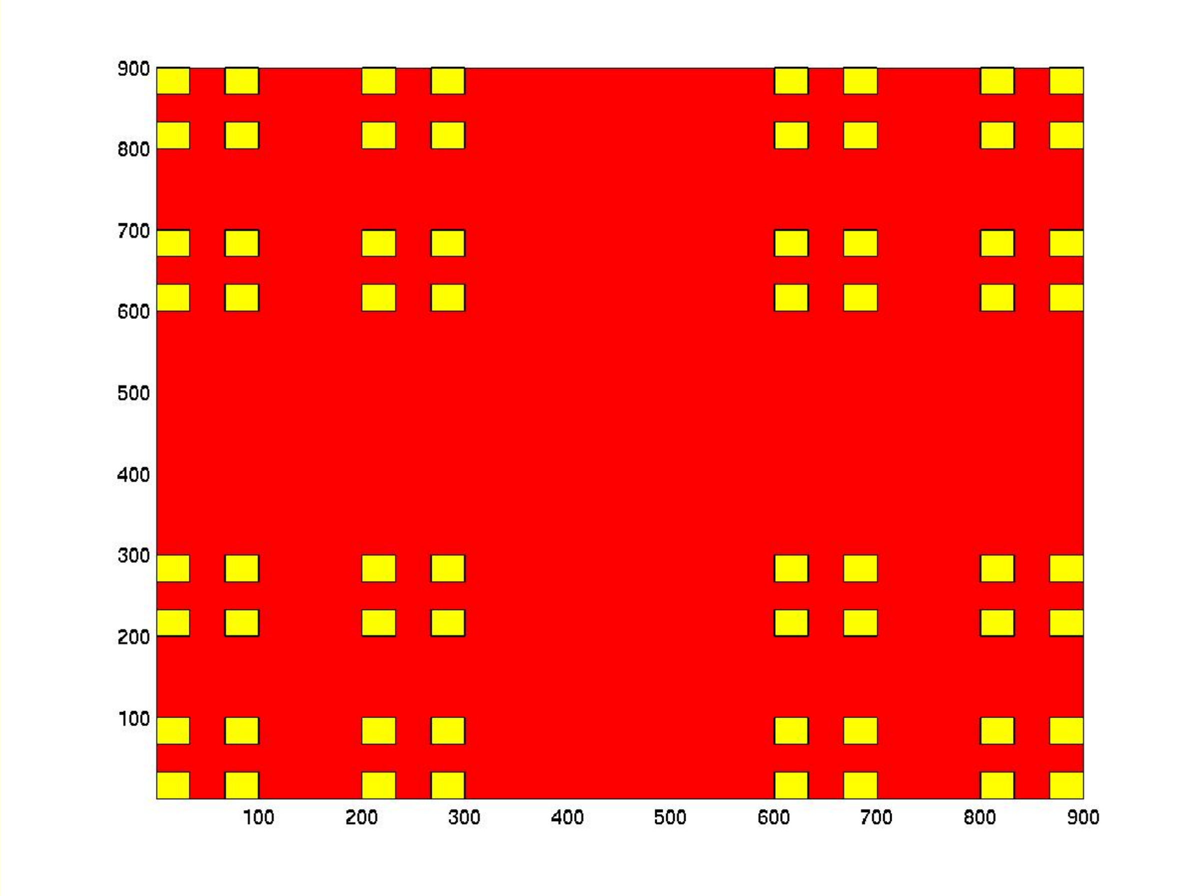
$$\Gamma_- = \bigcap_{N \geq 0} B^{-N}(\mathbf{T}^2) \subset \bigcap_{N=0,1,2} B^{-N}(\mathbf{T}^2)$$

And, finally the "trapped set" $K = \Gamma_+ \cap \Gamma_-$:



$$K = \Gamma_+ \cap \Gamma_- = \bigcap_{N \in \mathbf{Z}} \pi_L(B^N) \subset \bigcap_{|N|=1,2,3} \pi_L(B^N)$$

And, finally the "trapped set" $K = \Gamma_+ \cap \Gamma_-$:



Rectangular Smale horse shoe structure.

$$\dim K = 2\dim\left(\Gamma_- \cap W_{\mathrm{u}}\right) = 2\frac{\log 2}{\log 3}.$$

Quantization of the open Baker relation (Balazs-Voros, Saraceno-Vallejos).

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \mathcal{F}_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_{N/3} \end{pmatrix} .$$

where $\mathcal{F}_M$ is the discrete Fourier transform:

$$(\mathcal{F}_M)_{kl} = M^{-\frac{1}{2}} \exp(2\pi i k l / M) , \quad 0 \leq k, l \leq M - 1 .$$

There is a precise (mathematically rigorous) way of stating what it means to quantize a general symplectic relation.

$h \to 0$	$N \to \infty$
$\exp(-it(-h^2\Delta + V)/h)$	$B_N^t\,, \quad t = 0, 1, \cdots$
$e^{-itz/h}$ z a resonance of H	λ^t λ an eigenvalue of B_N
$z \in [E-h, E+h] - i[0, \gamma h]$	$\lvert\lambda\rvert > \rho > 0$
$\#\{z \in [E-h, E+h] - i[0, \gamma h]\}$ $\simeq \ C(\gamma) h^{-\mu_E}$	$\#\{\lambda, \lvert\lambda\rvert > \rho\}$ $\simeq \ C(\rho) N^{\frac{\log 2}{\log 3}}$

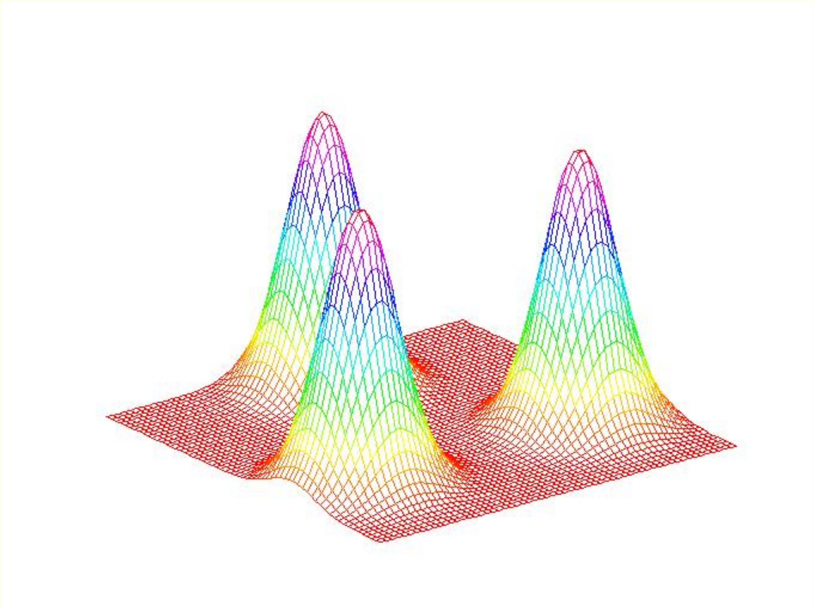
$$B_N = \mathcal{F}_N^* \begin{pmatrix} \mathcal{F}_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_{N/3} \end{pmatrix} , \quad 2\mu_E + 1 = \dim K_E \,.$$

Conjectural Fractal Weyl Law

$$\sharp\{\text{resonances of } -h^2\Delta + V \text{ in } D(E, rh)\ \} \sim C(r)h^{-\mu_E}\,,$$

$$\dim K_E = 2\mu_E + 1\,.$$

Here the potential is assumed to have a hyperbolic classical flow near energy E, for instance



and K_E is the trapped set at that energy.

Conjectural Fractal Weyl Law

$$\sharp\{\text{resonances of } -h^2\Delta + V \text{ in } D(E, rh)\} \sim C(r) h^{-\mu_E},$$

$$\text{dim} K_E = 2\mu_E + 1 .$$

Weyl Law for closed systems

$$\sharp\{\text{resonances of } -h^2\Delta + V \text{ in } D(E, rh)\} =$$

$$\frac{1}{(2\pi h)^n} \int_{|p-E| \leq rh} dx d\xi + o(h^{-n+1}) \sim C(r) h^{-n+1} .$$

When everything is trapped

$$\text{dim } K_E = \text{dim(energy surface)} = 2(n-1) + 1 .$$

Mathematical results:

Precise upper bounds (without good estimates on $C(r)$): Guillopé-Lin-Zworski 2003, Sjöstrand-Zworski 2005 (earlier work by Sjöstrand 1991 and Zworski 1999).

Numerical results:

Lin (J. Comp. Phys. 2002), Lin-Zworski (Chem. Phys. Lett. 2002): Quantum resonances for the three bumps potential.

Lu-Sridhar-Zworski (Phys. Rev. Lett. 2003). Resonances for three discs computed using the semi-classical zeta function (Cvitanovič, Eckhardt, Gaspard...).

Strain-Zworski (Nonlinearity 2004) Resonances for $z \mapsto z^2 + c$, $c < -2$ computed using a new method based on the upper bounds technology for zeta functions.

For B_N the Fractal Weyl law says:

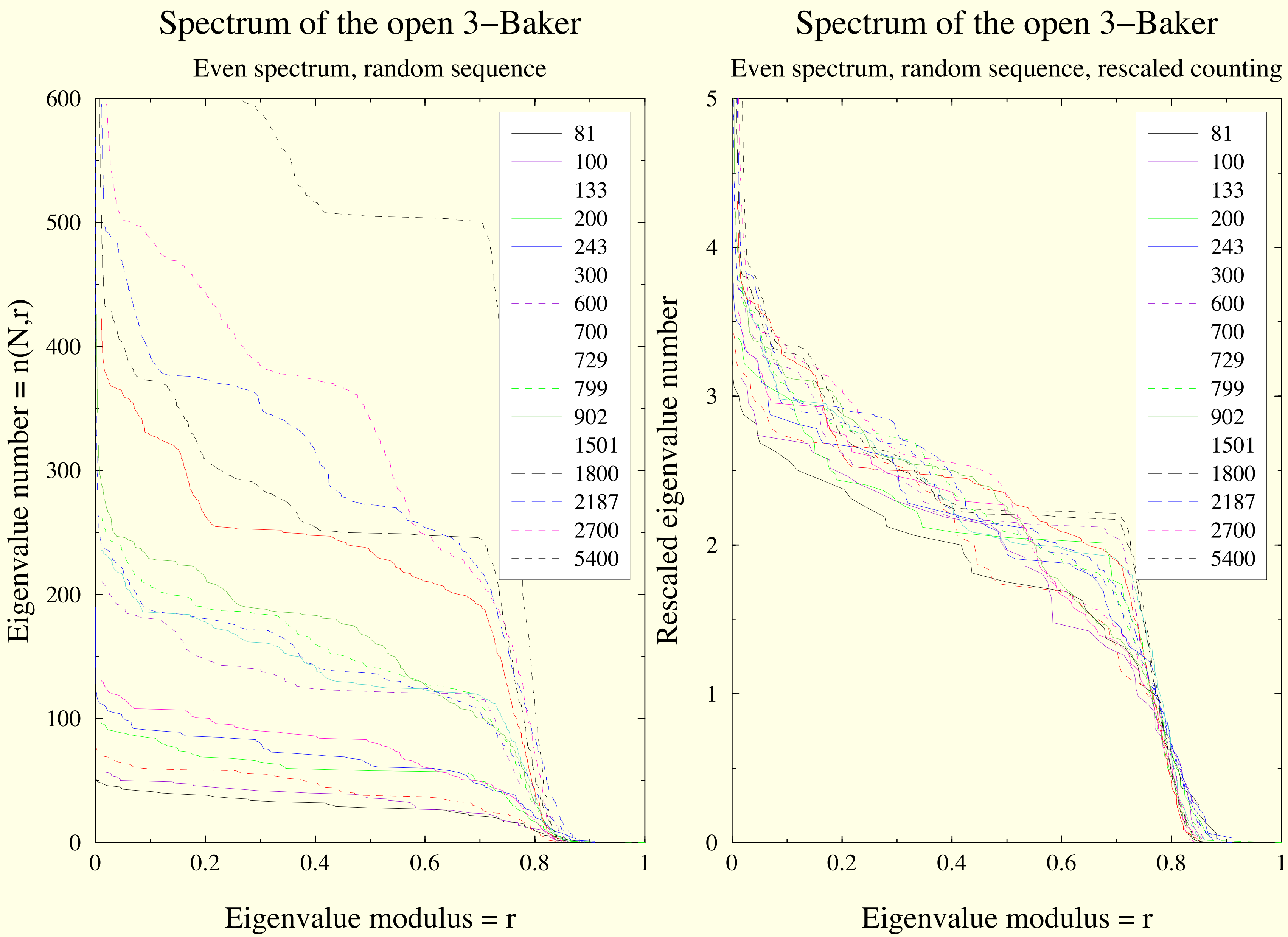
$$\sharp\{\text{eigenvalues of } B_N \text{ with } |\lambda| > r\} \sim C(r) N^{\mu}\,.$$

$$\mu = \frac{1}{2}\dim K = \dim\left(\Gamma_- \cap W_{\mathrm{u}}\right) = \frac{\log 2}{\log 3}\,.$$

Numerical evidence supports this conjecture.

Similar evidence was recently obtained by Schomerus-Tworzydło (Phys. Rev. Lett. 2004) for the open quantum kicked rotor.

To illustrate our data we follow Schomerus-Tworzydło:



On the right we see (?) the hypothetical function $C(r)$.

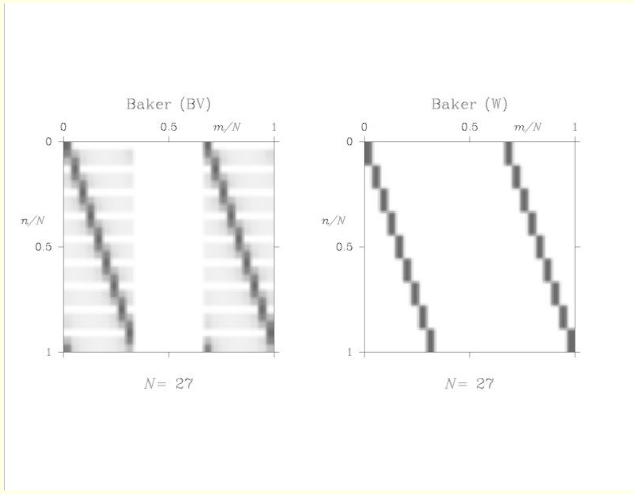
Computable Toy Model

We form a matrix $\widetilde{B}_N$ by keeping the "most significant elements" of B_N:

$$\widetilde{B}_9 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix}, \ \omega = e^{2\pi i/3}$$

A computable Toy Model

We form a matrix $\widetilde{B}_N$ by keeping the "most significant elements" of B_N:

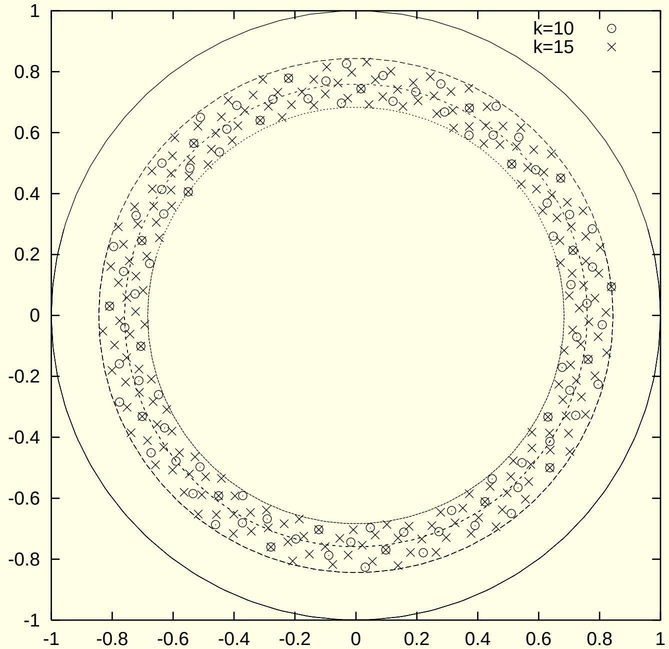


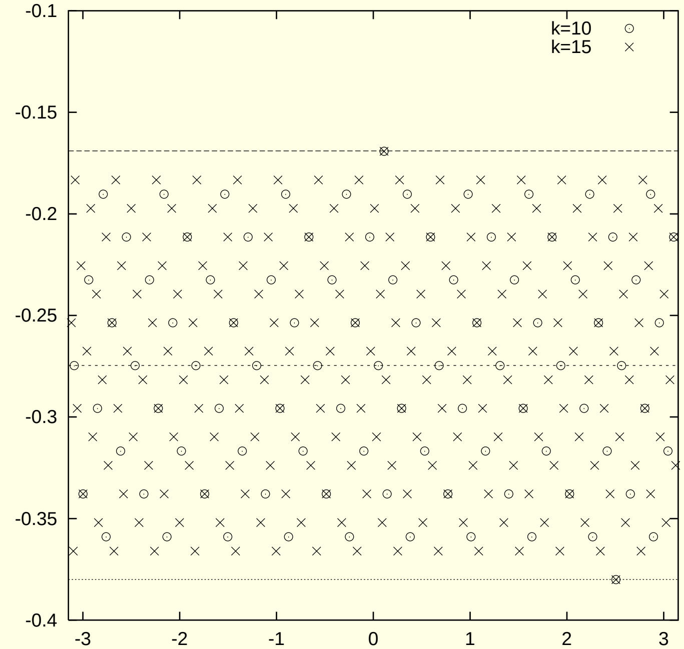
$\widetilde{B}_N$ has been proposed before as a "toy quantization" of the open Baker map (Schack-Caves, Saraceno). It also appears in the study of quantum binary graphs (Tanner).

It is perhaps a bit surprising that $\widetilde{B}_N$ is a quantization of a more complicated classical relation and one for which we still have $\dim \Gamma_- \cap W_{\mathrm{u}} = \log 2 / \log 3$.

The Fractal Weyl law holds exactly for the toy model when $N = 3^k$:

$$\sharp\{\text{eigenvalues of } B_{3^k} \text{ with } |\lambda| > r\} = (C(r) + O(1/k))2^k\,.$$





Why can we compute it (almost) exactly?

$$\widetilde{B}_{3^k} = W_k^* \begin{pmatrix} W_{k-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{k-1} \end{pmatrix},$$

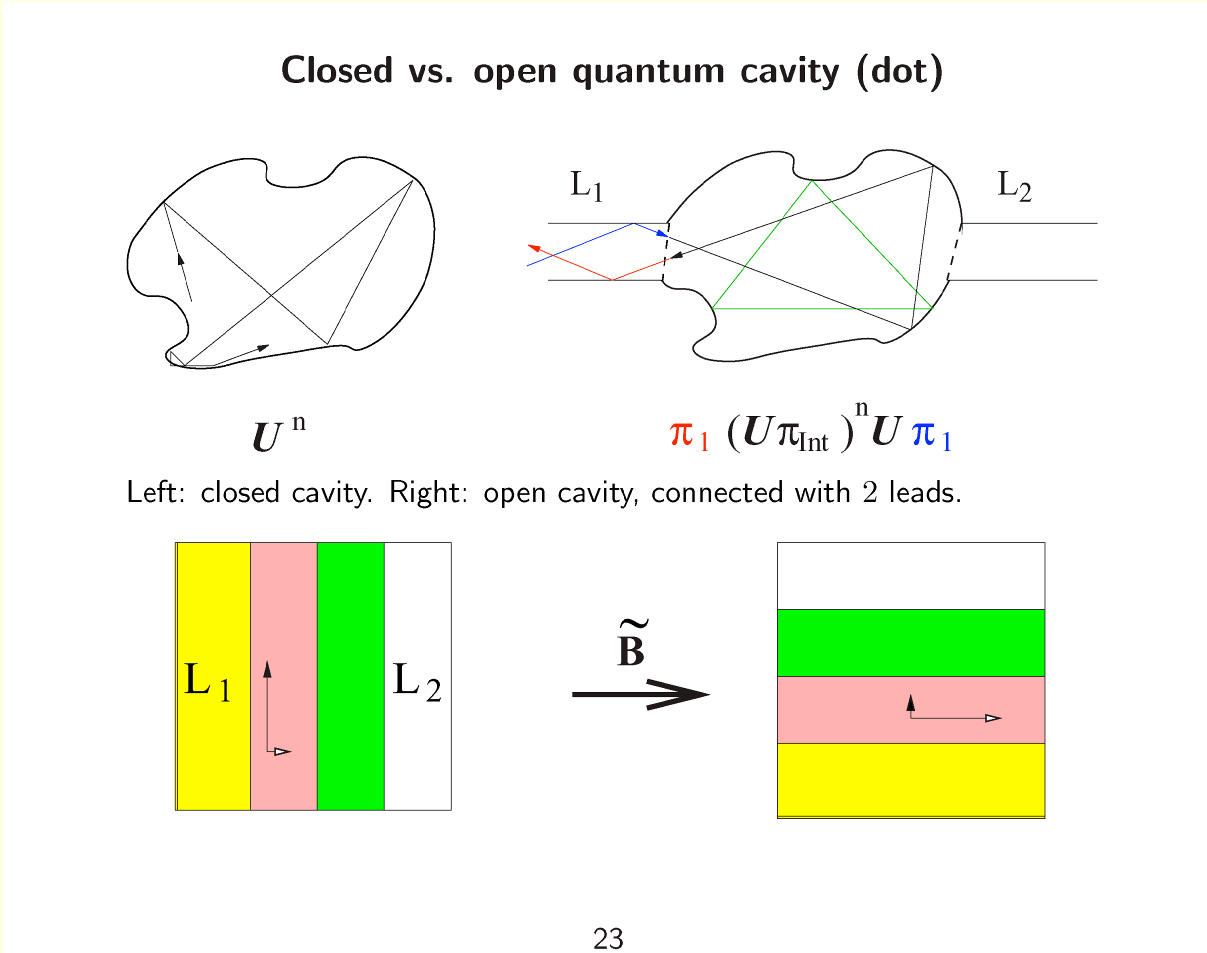
where W_k is the Walsh Fourier transform which is the Fourier transform on the group $(\mathbf{Z}_3)^k$, rather than, as $\mathcal{F}_{3^k}$, on $\mathbf{Z}_{3^k}$.

Functions on $(\mathbf{Z}_3)^k$ (our Hilbert space of dimension 3^k) are identified with $(\mathbf{C}^3)^{\otimes k}$ and the action of W_k is very simple:

$$W_k(v_1 \otimes \cdots \otimes v_k) = (W_1 v_k \otimes \cdots \otimes W_1 v_1),$$

$$\widetilde{B}_{3^k}(v_1 \otimes \cdots \otimes v_k) = (v_2 \otimes \cdots \otimes v_k \otimes W_1 v_1).$$

The toy model can be used to compute other quantities:



The scattering matrix is given by Miller's formula:

$$S_N(\theta) = (\pi_1 + \pi_2) \sum_{k \geq 0} (e^{-i\theta} U(I - \pi_1 - \pi_2))^k e^{i\theta} U(\pi_1 + \pi_2) \,.$$

Transmission part of S:

$$t_{12}(\theta) = \pi_1 \sum_{k\geq 0} (e^{-i\theta} U(I - \pi_1 - \pi_2))^k e^{i\theta} U \pi_2 \,.$$

Conductance $\sim \mathrm{tr}\, t_{12} t_{12}^*$

Shot Noise $\sim \mathrm{tr}\, t_{12} t_{12}^* (I - t_{12} t_{12}^*)$

Weidenmüller, Blümel-Smilansky, Beenakker...

In the toy model:

$$\mathrm{tr}\ t_{12}t_{12}^* = \frac{4^{k-1}}{2}(1+2^{-\alpha k})$$

$$\mathrm{tr}\ t_{12}t_{12}^*(I - t_{12}t_{12}^*) = 2^{k-1}\frac{11}{80}(1+2^{-\alpha k})\,.$$

The last expression indicates that the "fractal Weyl law" appears in the shot noise, $2^{k-1} = N^{\mu}/2$.

The random matrix theory prediction (Beenakker et al), once corrected by the fractal Weyl law gives the factor $1/8 \simeq 11/80$.

Even in a computable non-generic model this seems remarkably close!