Schloss Reisensburg

What are the residues of the resolvent of the Laplacian on non-compact symmetric spaces?

Maciej Zworski

UC Berkeley

25 July 2006

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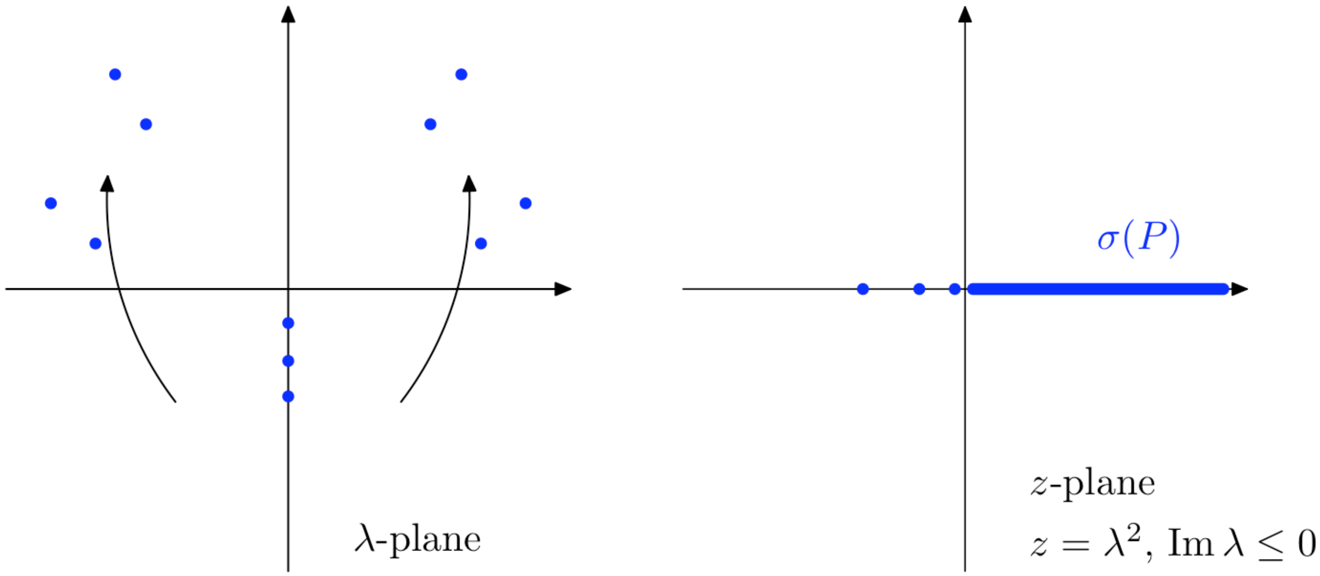

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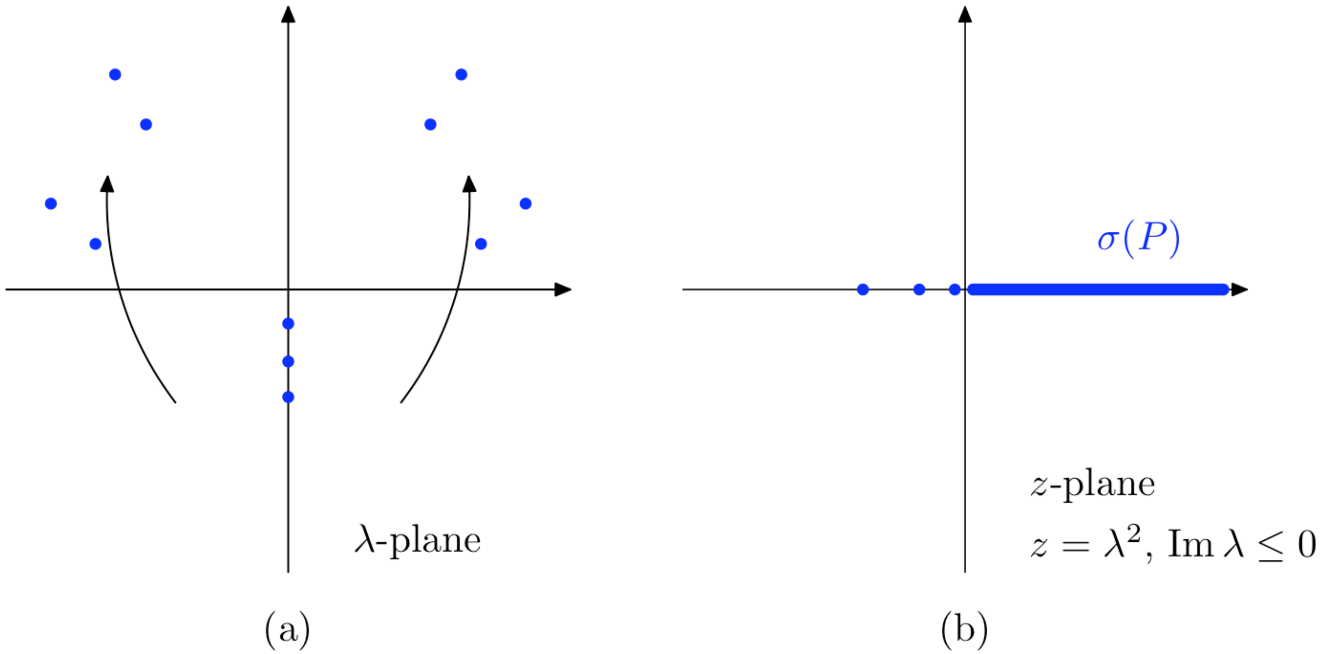

(a) Meromorphic continuation of the resolvent $R(\lambda) = (P - \lambda^2)^{-1}$; the poles in the lower half-plane correspond to negative eigenvalues of P.

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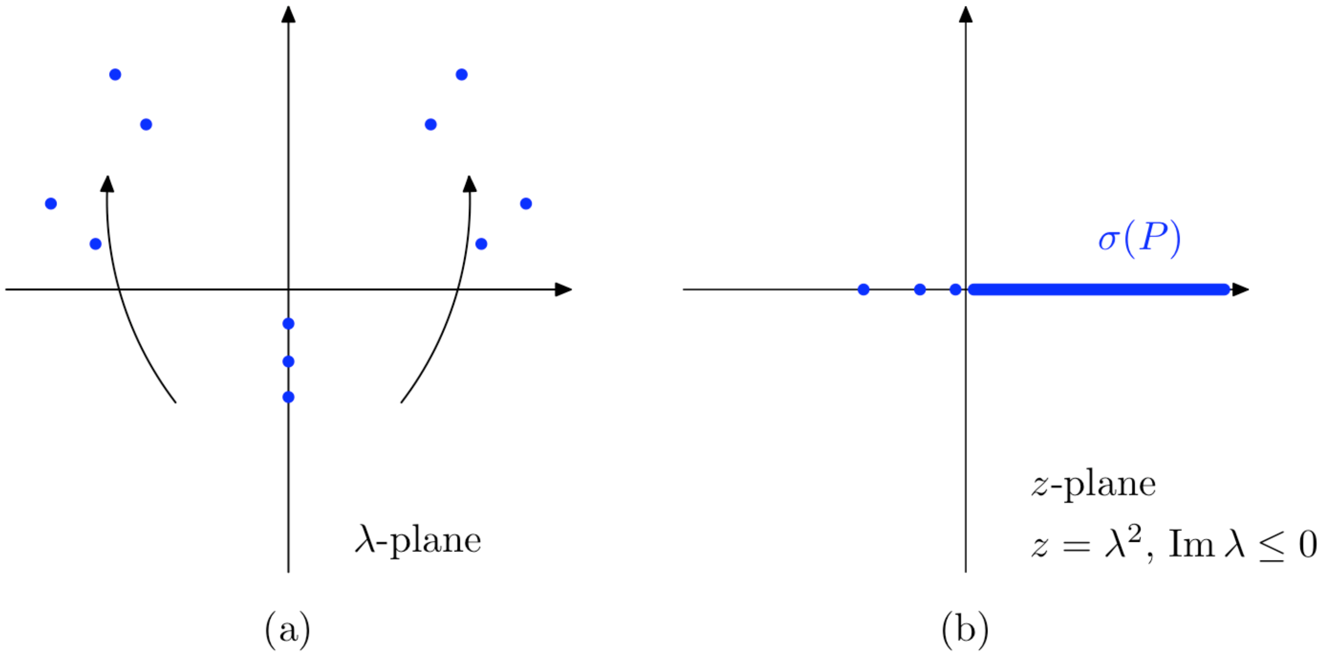

(b) The spectrum of P: the z-plane, $z = \lambda^2$.

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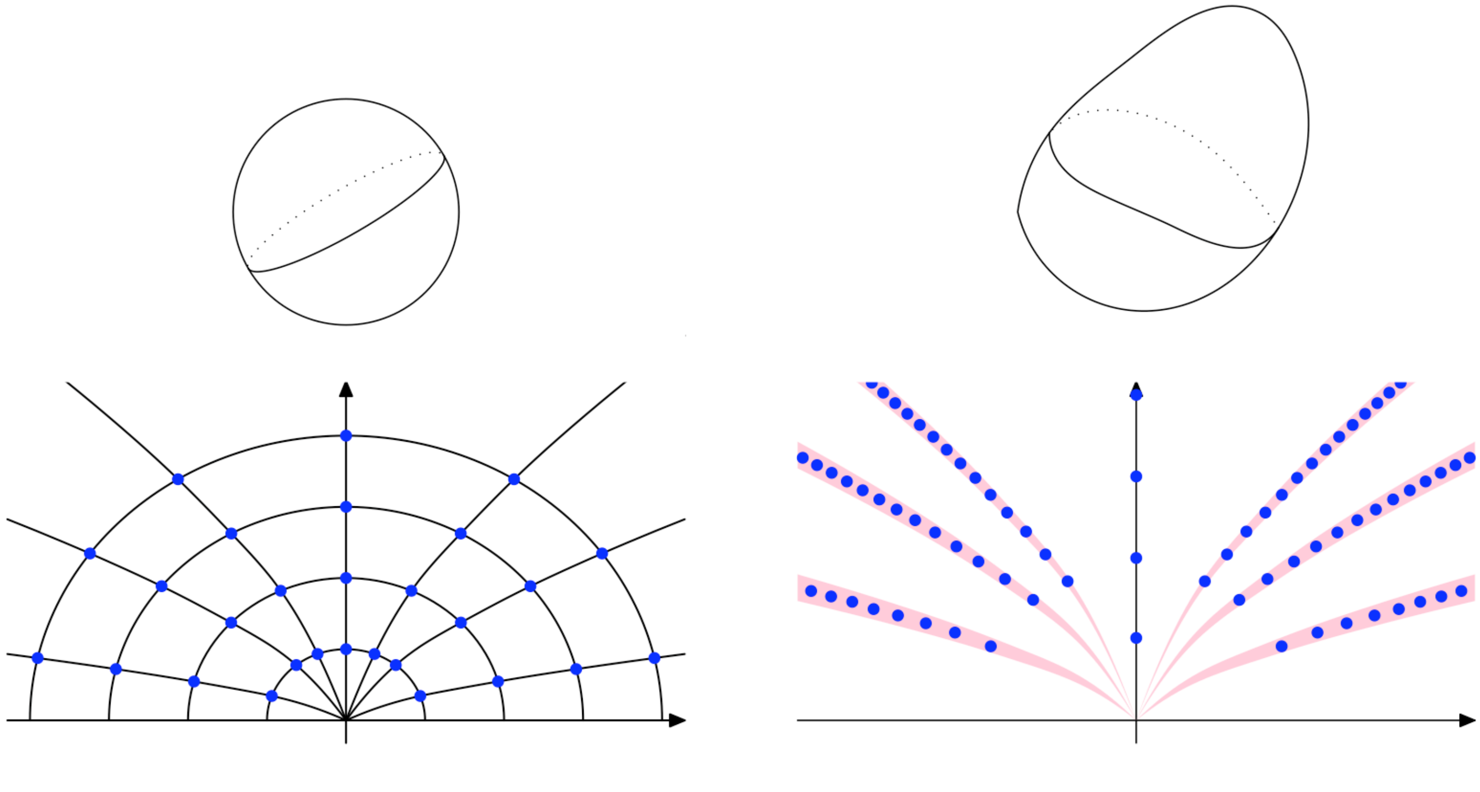

(a)

(b)

(a) Resonances for the sphere; the resonances on the lth ring have multiplicity $2l+1$ (Watson 1918).

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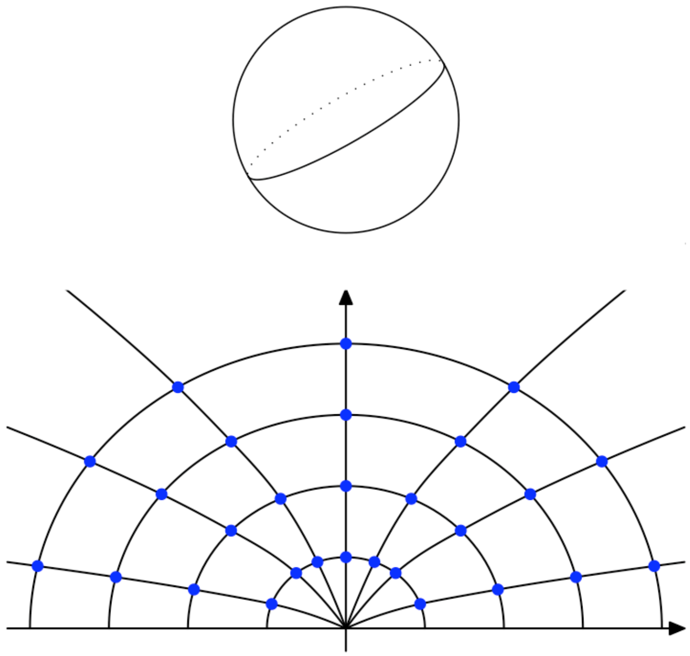
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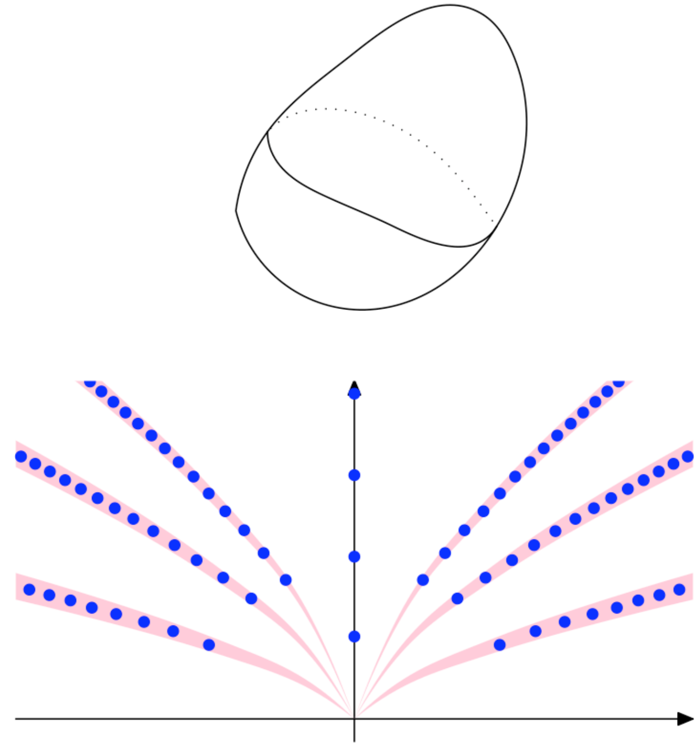
(b)

(b) Resonances for a convex body, $\mathcal{O}$, satisfying the pinched curvature assumption. The resonances in each band (on a cubic curve in case of the sphere) satisfy the same Weyl law as the eigenvalues of the Laplacian on $\partial\mathcal{O}$ (Sjöstrand-Z 1999).

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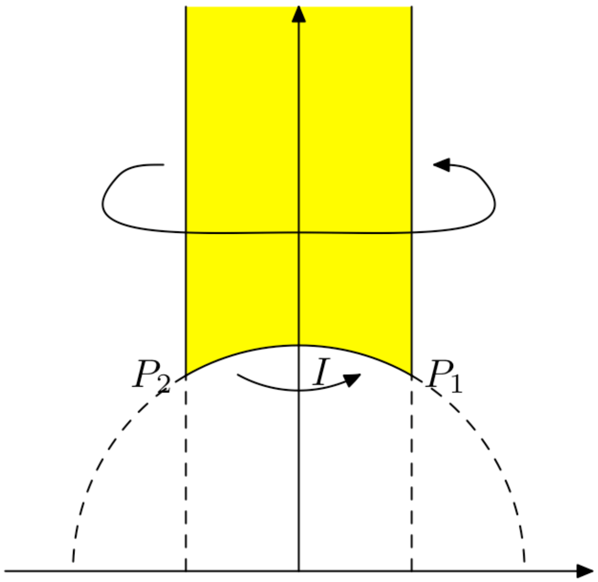

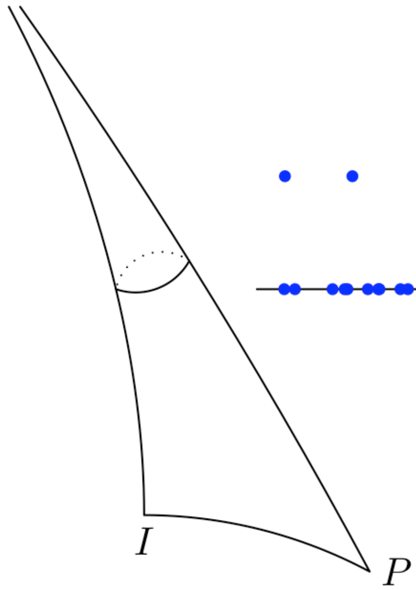

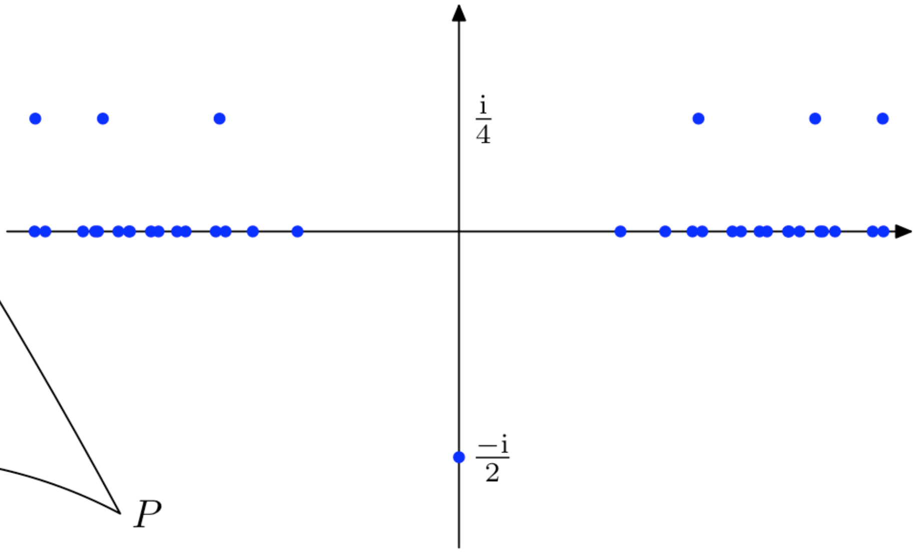

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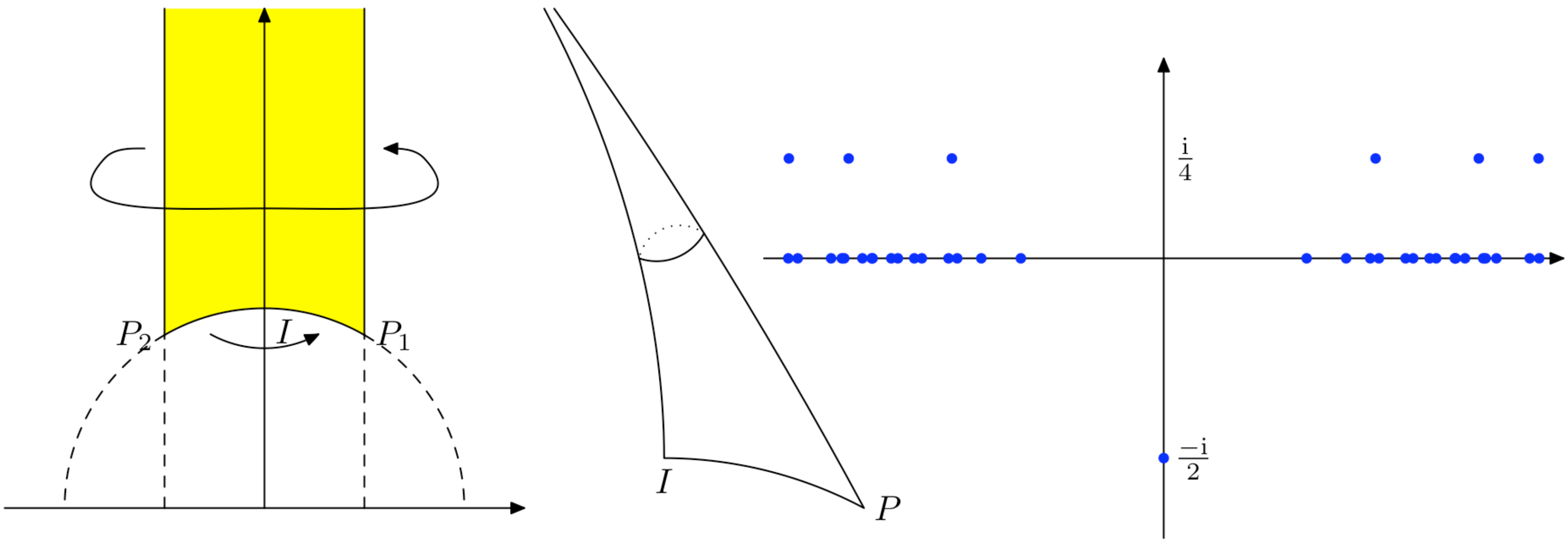

The fundamental domain of the modular group and the resonances of the modular surface (Faddeev-Pavlov 1972, Lax-Phillip 1976)

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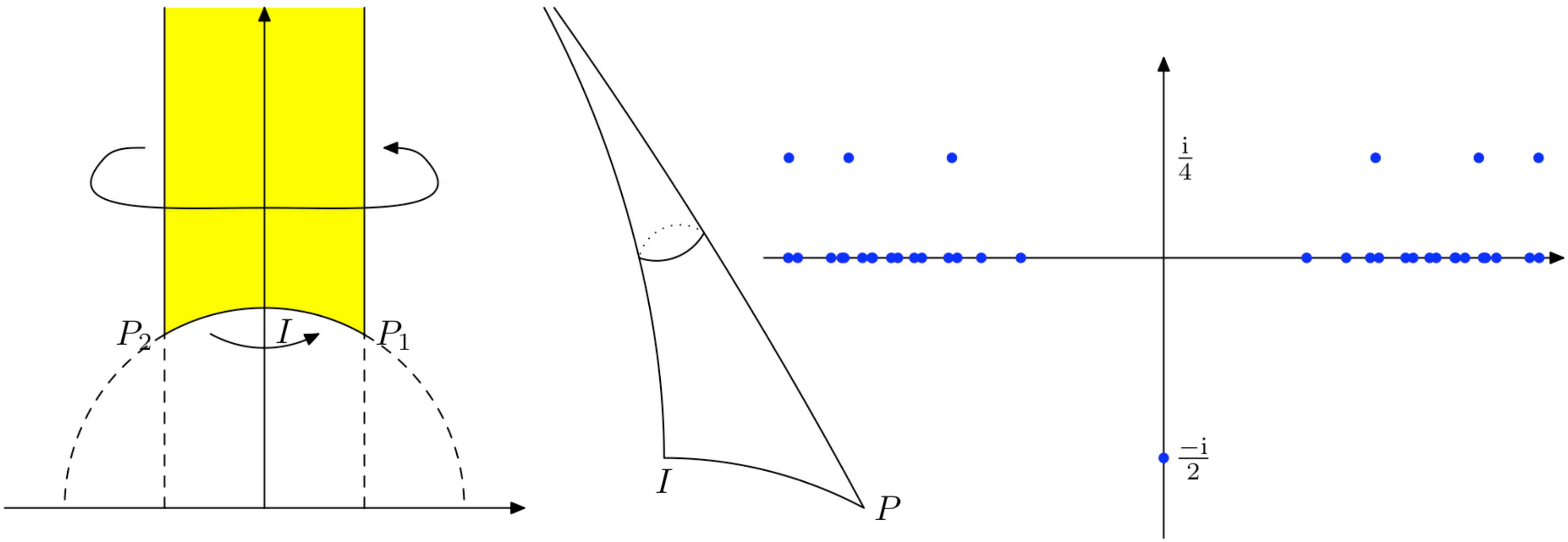

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This figure assumes the Riemann hypothesis!

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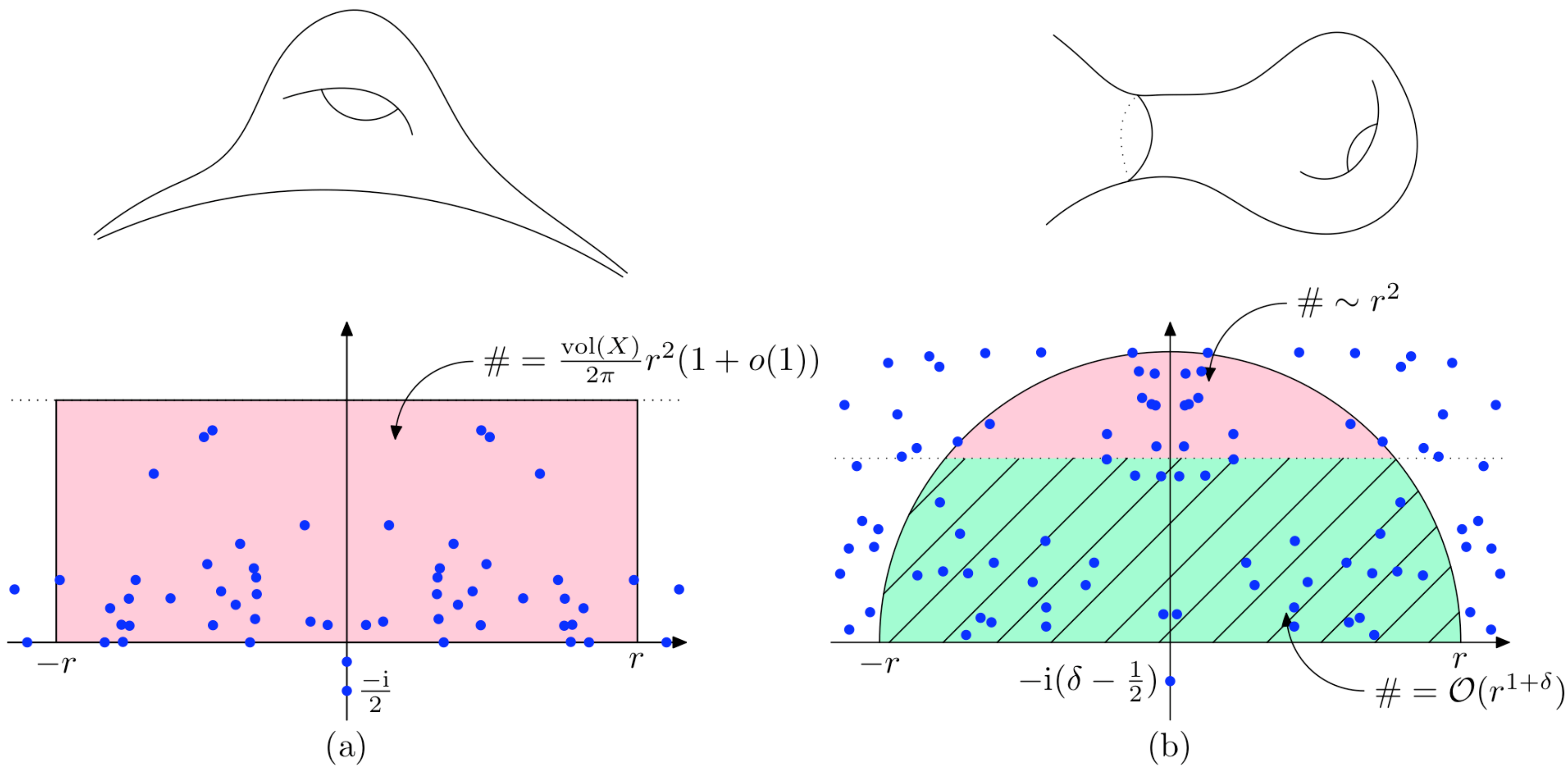

$\# = \frac{\mathrm{vol}(X)}{2\pi} r^2(1+o(1))$
$-r$
r
$\frac{-\mathrm{i}}{2}$
(a)
$\# \sim r^2$
$-r$
r
$-\mathrm{i}(\delta - \frac{1}{2})$
$\# = \mathcal{O}(r^{1+\delta})$
(b)

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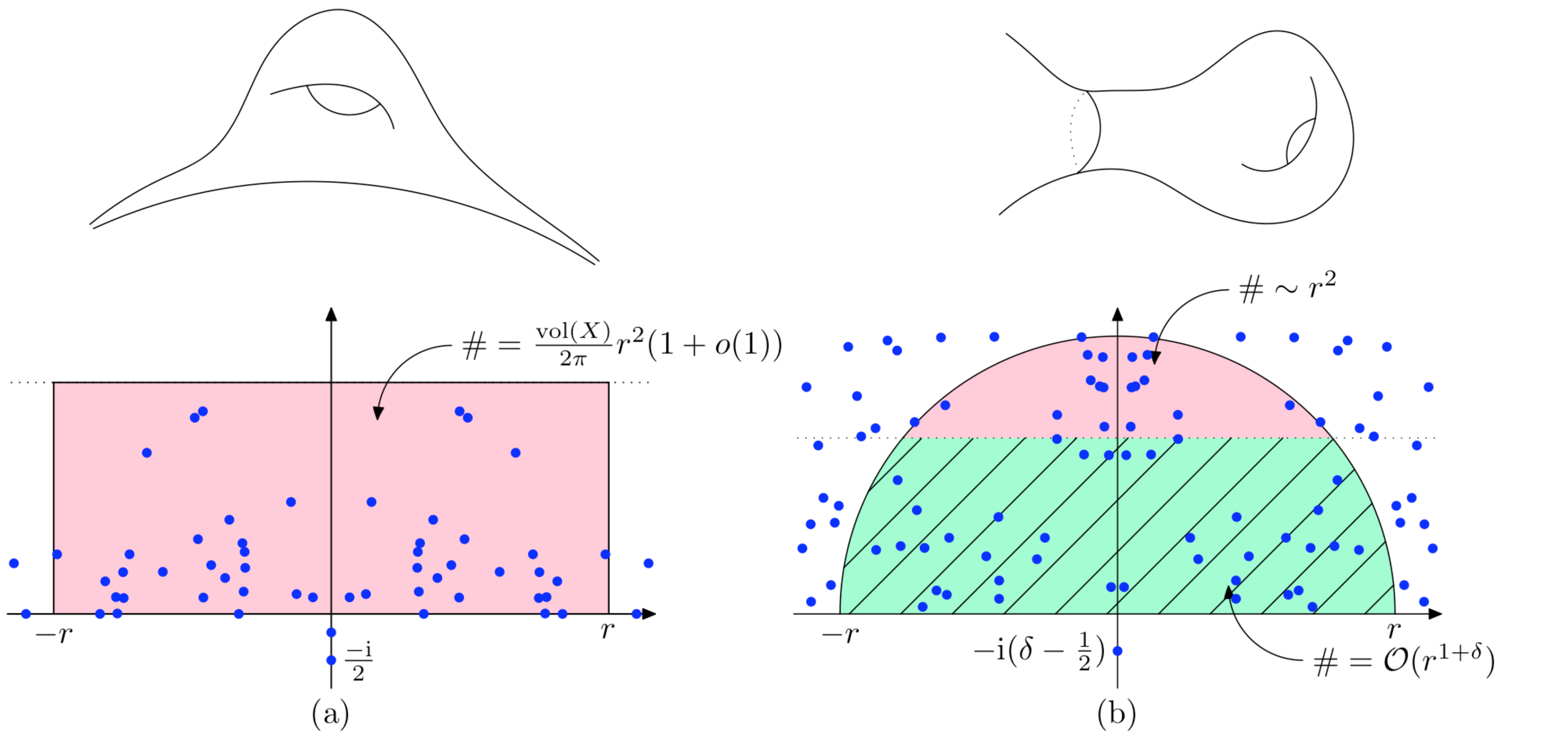

(a) Resonances for a finite volume hyperbolic surface: they are confined to a horizontal strip and satisfy the usual Weyl law (Selberg 1954, Müller 1992).

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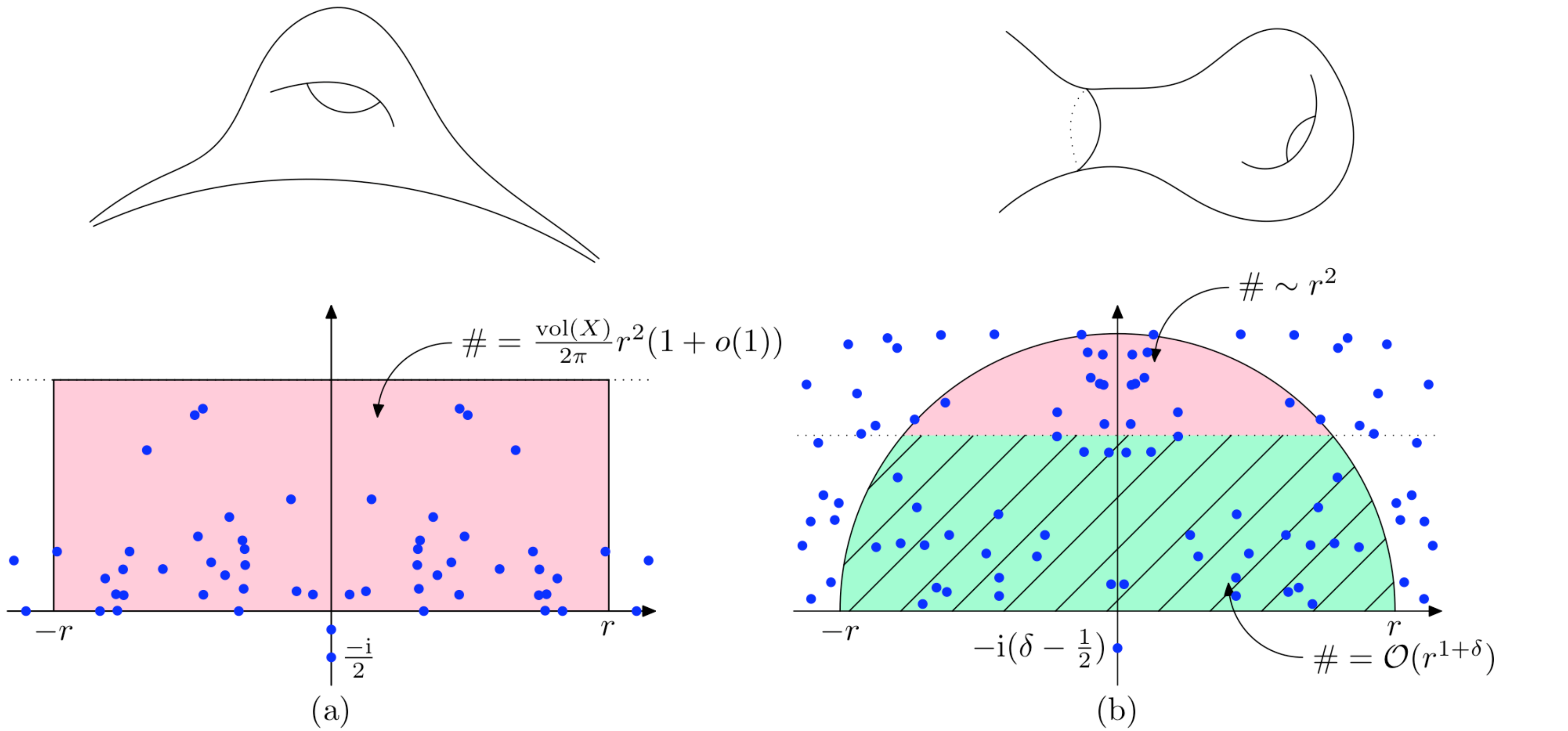

(b) Resonances for an infinite volume surface with no cusps: they are scattered all over the upper half-plane; the counting function is bounded from above and below by multiples of r^2 (Guillope-Z 1997).

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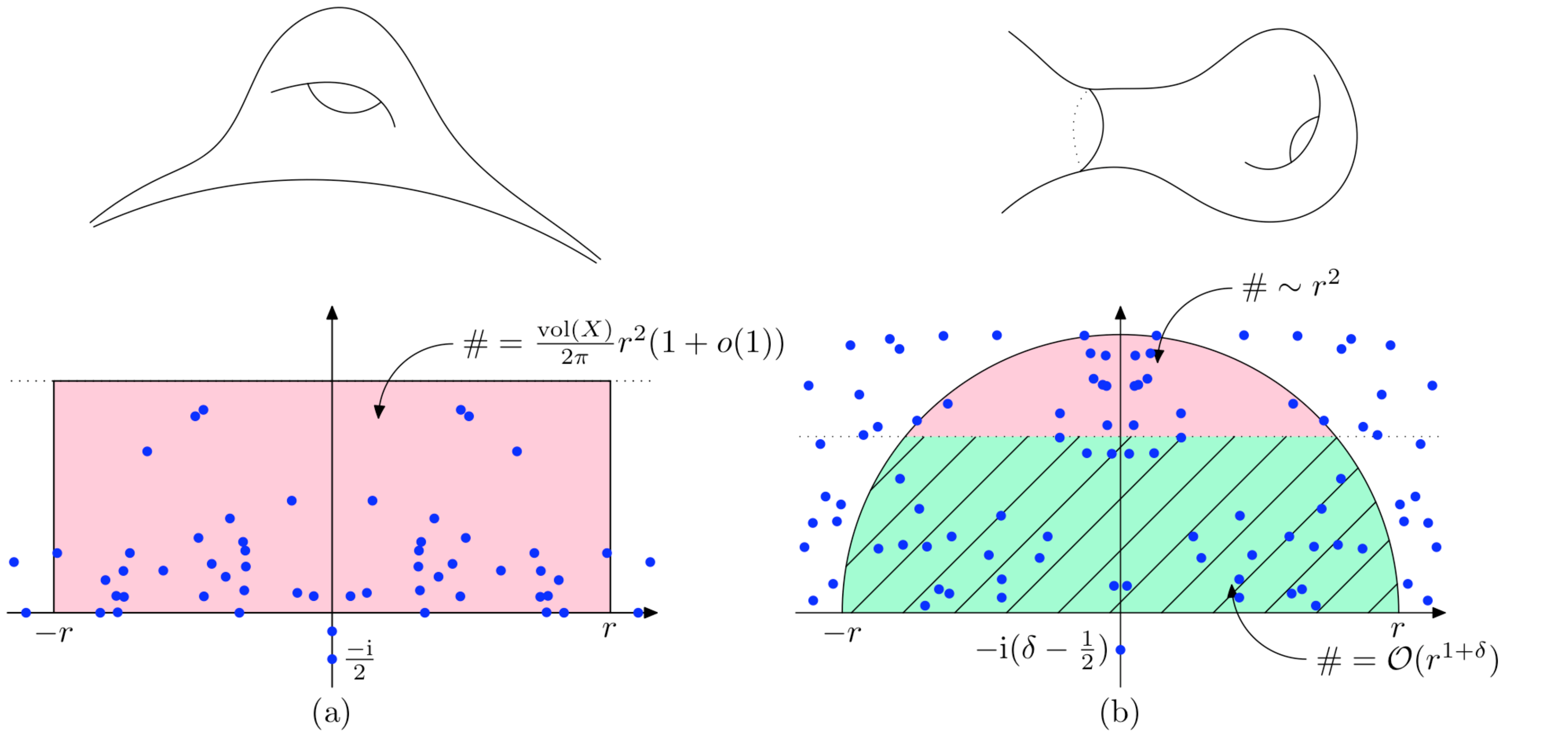

(b) When we count only in a strip the number of resonances is bounded by a multiple of $r^{1+\delta}$, where δ is the dimension of the limit set (Z 1999, Guillopé-Lin-Z 2004).

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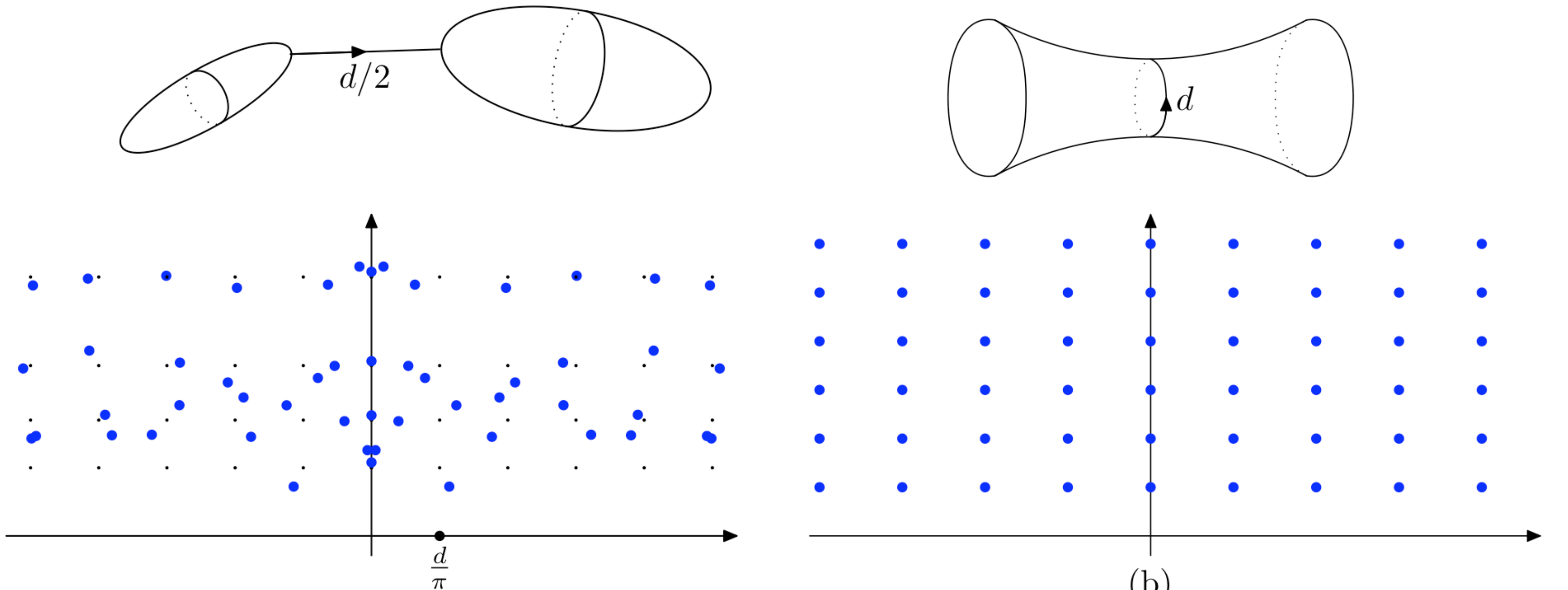

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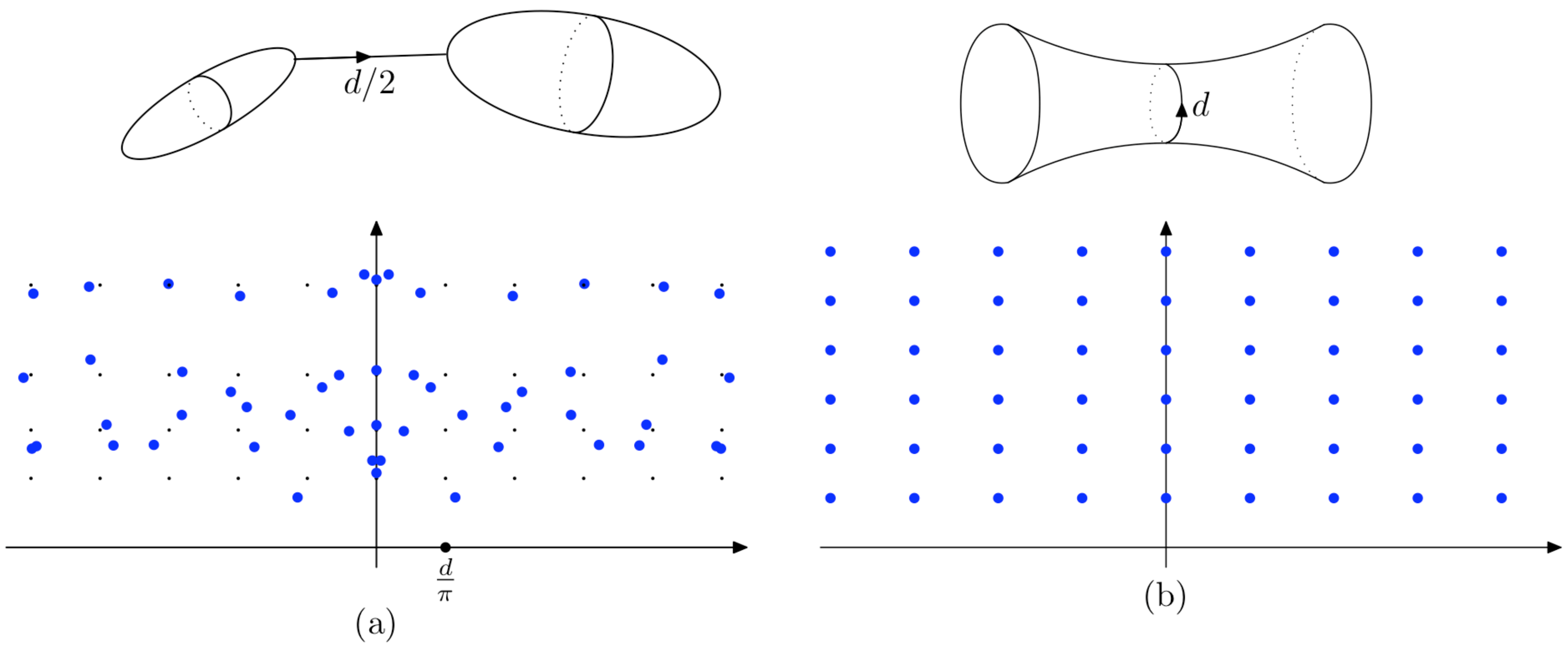

(a) Resonances associated to two strictly convex bodies: in every fixed strip, the resonances become closer to points on the lattice as the real part increases (Ikawa 1986, Gérard 1988).

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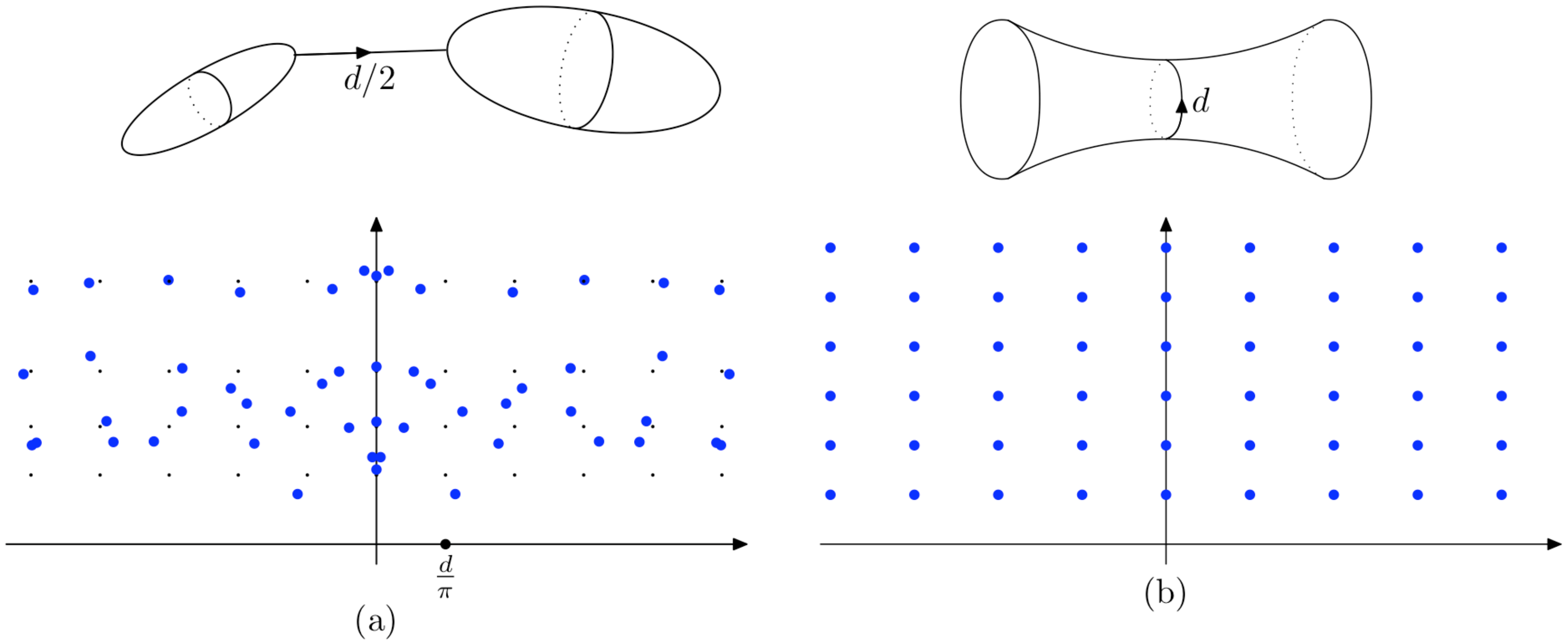

(b) Resonances for a hyperbolic cylinder: all resonance lie exactly on a lattice. The underlying dynamical structure, exactly one hyperbolic closed orbit, is the same in the two examples (Guillopé 1988).

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The spaces $L^2(\mathbf{H}^2)$, $H^s(\mathbf{H}^2)$ are considered with the measure $(1 - |z|^2)^{-2}\mathcal{L}(dz)$ where $\mathcal{L}(dz)$ is the Euclidean measure on $\mathbf{C}$.

The spectrum of the Laplacian is absolutely continuous and equal to $[1, \infty]$ so that the resolvent $(\Delta - s(2-s))^{-1}$ is bounded on $L^2(\mathbf{H}^2)$ for $\mathrm{Re}\, s > 1$. We want to show that

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We should stress that the meromorphic continuation of the resolvent is well known. Howover, rather than to relay on special functions and explicit formulæ we want to use analysis on symmetric spaces.

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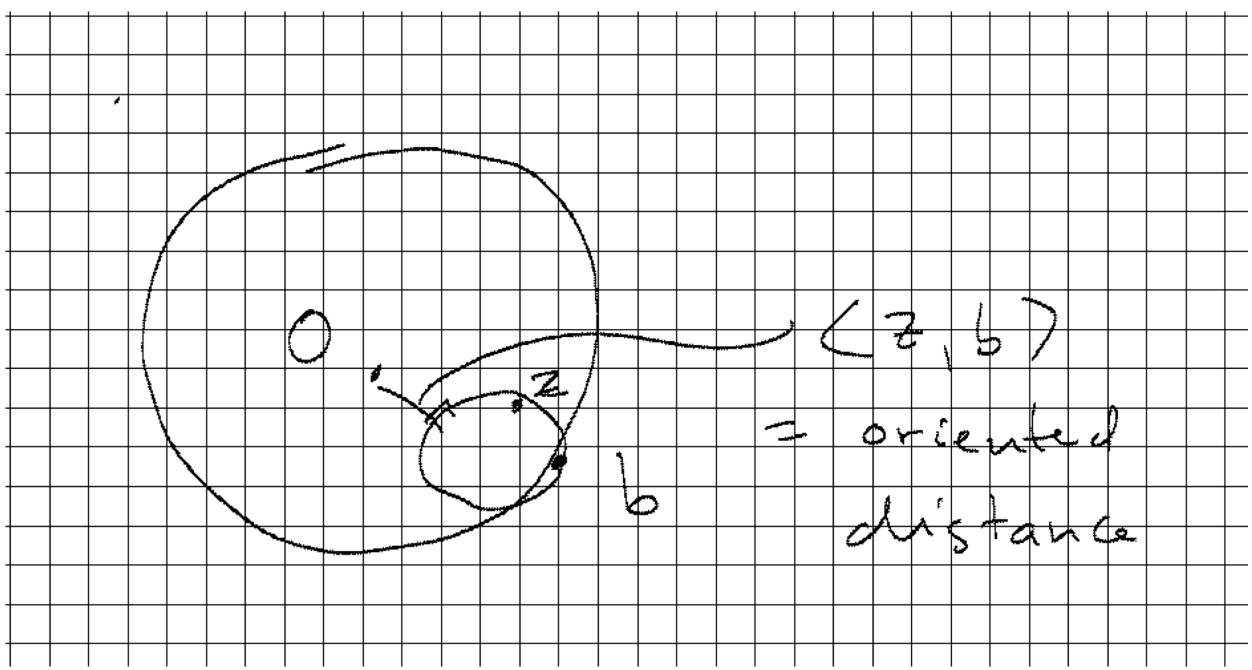

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$$G(\lambda, z, w) = G(\lambda, gz, gw)\,, \quad g \in SU(1,1)\,,$$

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the spherical function $\phi_\zeta(z)$ is entire in ζ for all $z \in \mathbf{H}^2$.

To continue analytically from $\operatorname{Im}\lambda > 0$ we will now deform the contour of the ζ-integration in the expression for $G(\lambda, z, 0)$:

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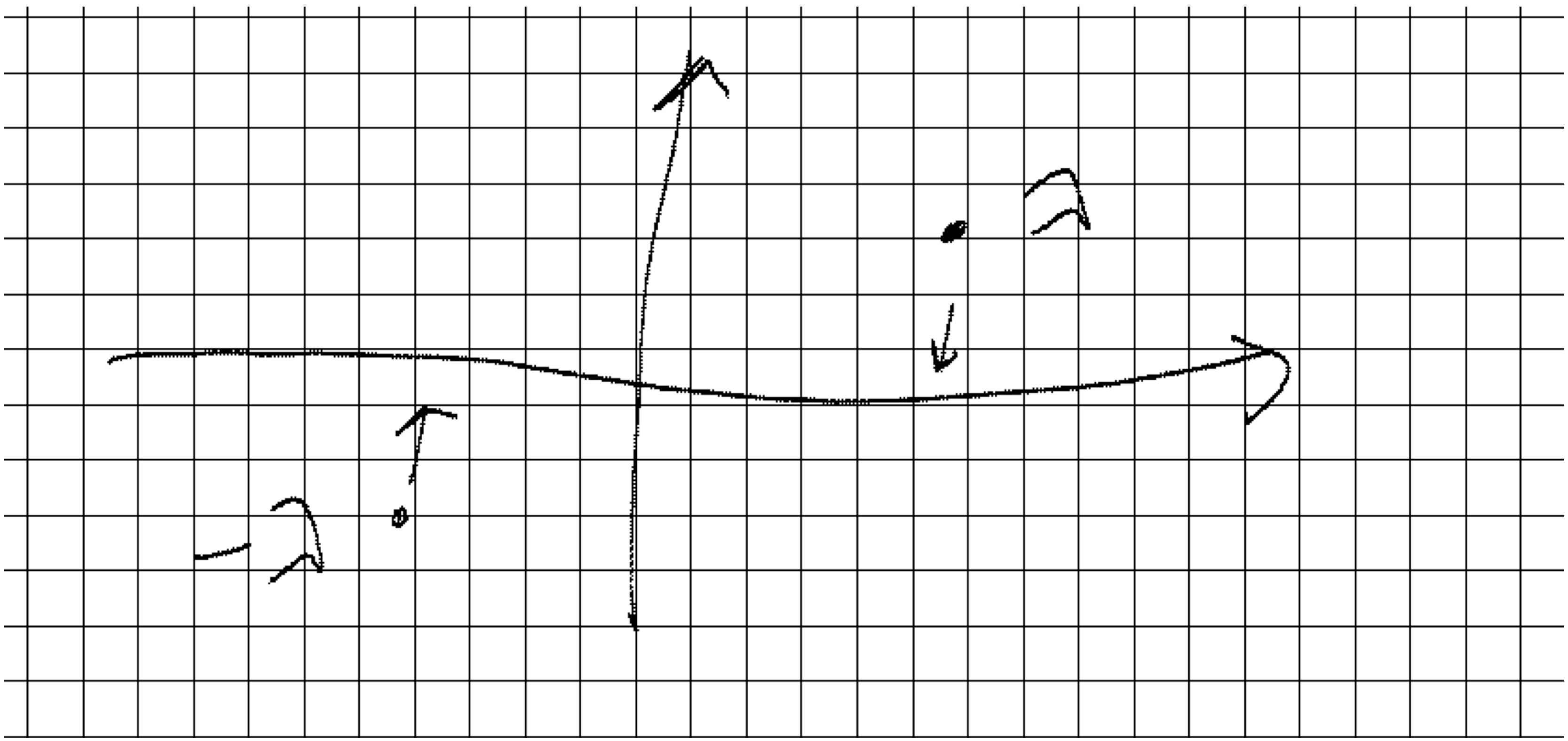

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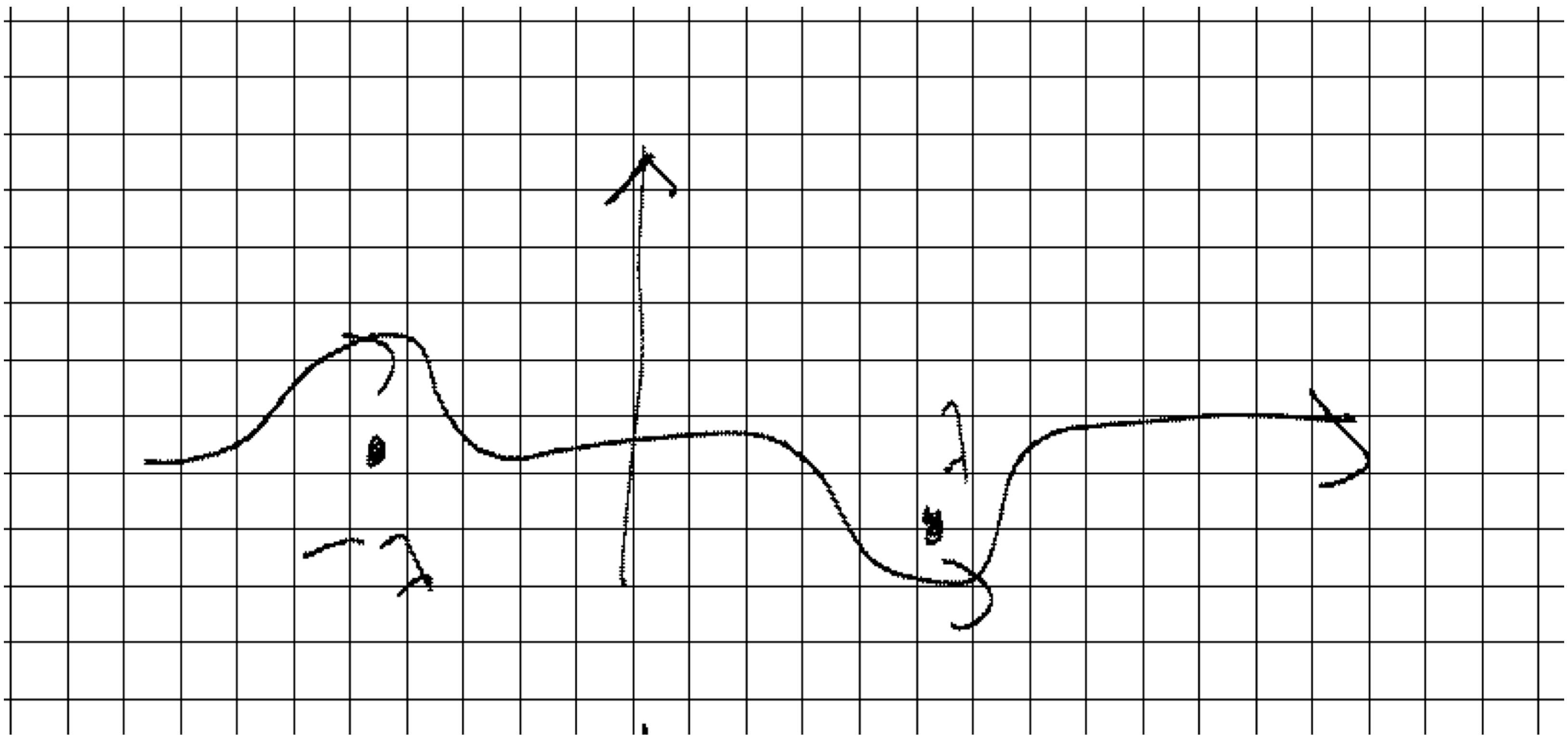

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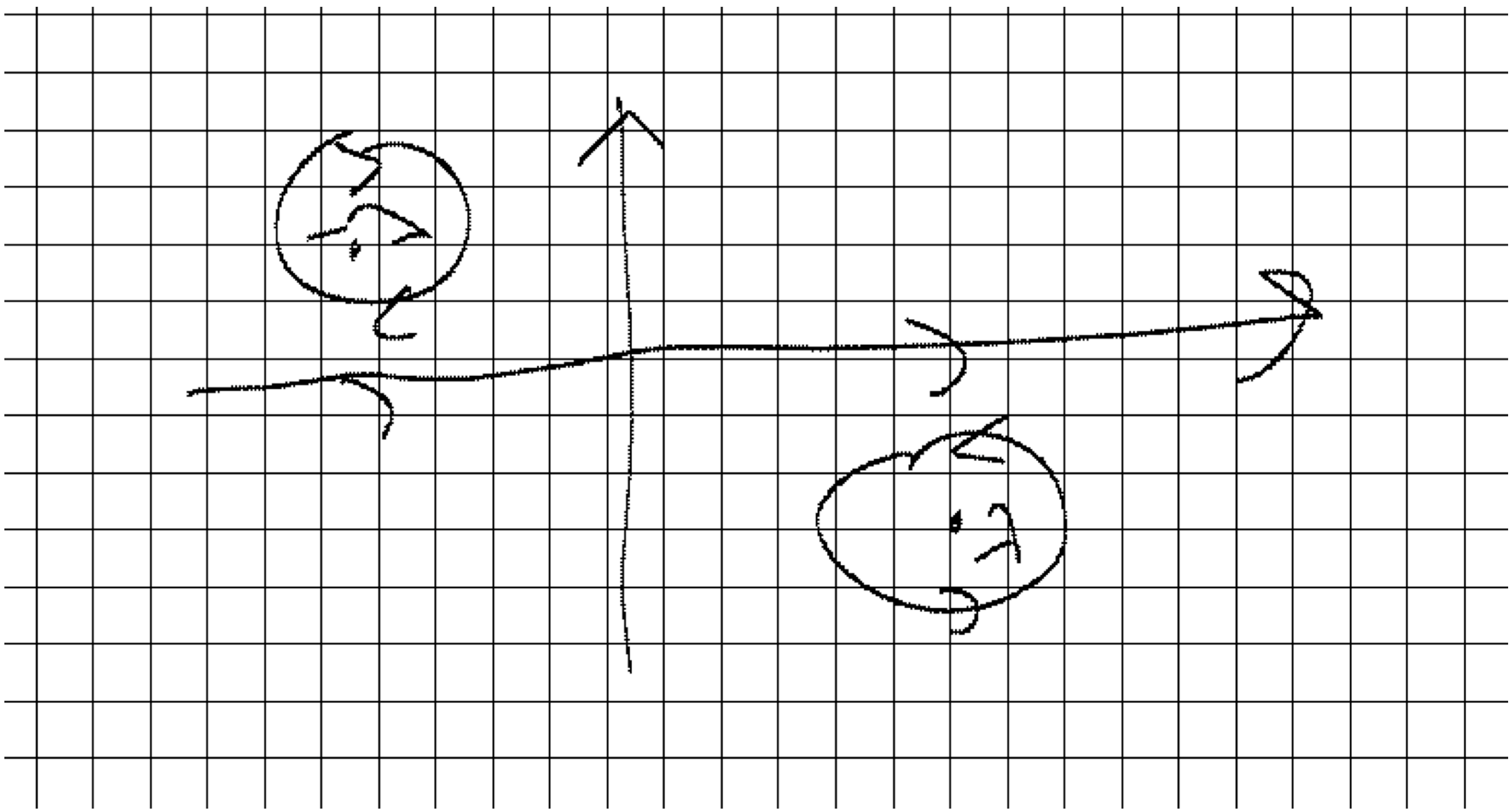

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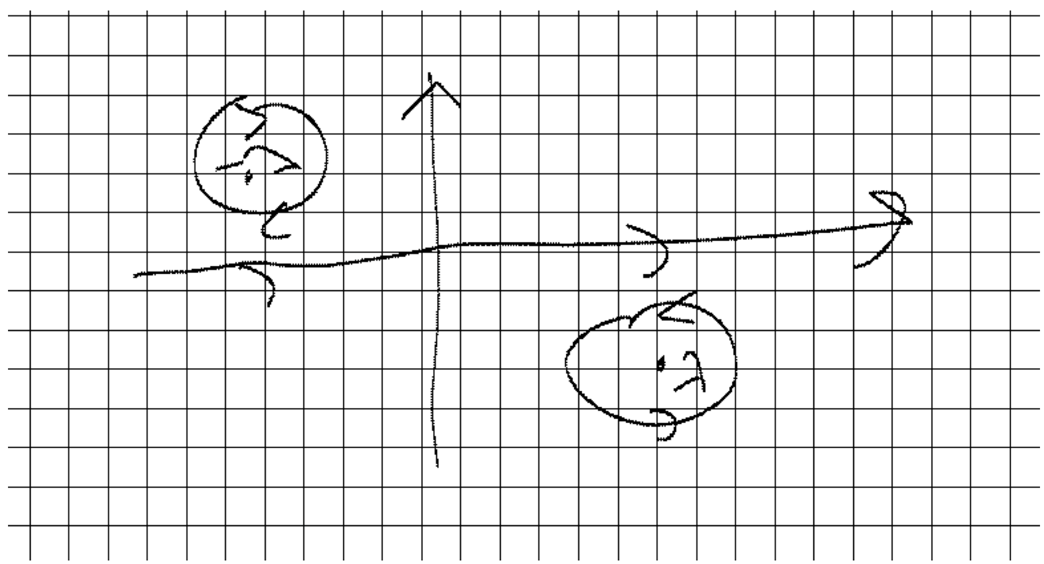

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