SEMICLASSICAL RESOLVENT ESTIMATES IN CHAOTIC SCATTERING

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1. Statement of Results

In this short note we prove a resolvent estimate in the pole free strip for operators whose classical Hamiltonian flows are hyperbolic on the trapped sets which are assumed to be sufficiently filamentary – see (1.4) for the precise condition. The proof is based on the arguments of [17] and we refer to §3 of that paper for the preliminary material and assumptions on the operator.

The polynomial estimate on the resolvent in the pole free strip (1.5) provides a direct proof of the estimate on the real axis (1.6), and that estimate is only logarithmically weaker than the bound in the non-trapping case. Through an argument going back to Kato, and more recently to to Burq, that estimate is crucial for obtaining local smoothing and Strichartz estimates for the Schrödinger equation. These in turn are important in the investigation of nonlinear waves in non-homogeneous trapping media. Also, as has been known since the work of Lax-Phillips, the estimate in the complex domain is useful for obtaining exponential decay of solutions to wave equations (see the paragraph following (1.6) for some references to recent literature).

An example of an operator to which our methods apply is given by

$$Pu = P(h)u = -h^2 \frac{1}{\sqrt{\bar{g}}} \sum_{i,j=1}^{n} \partial_{x_j} \left(\sqrt{\bar{g}} g^{ij} \partial_{x_i} u \right) + V(x) \,, \quad x \in \mathbb{R}^n \,, \tag{1.1}$$

$G(x) \stackrel{\text{def}}{=} (g^{ij}(x))_{i,j}$ is a symmetric positive definite matrix, $\bar{g} \stackrel{\text{def}}{=} 1/\det G(x)$, and

$$g^{ij}(x) = \delta_{ij} \,, \quad V(x) = -1 \,, \quad |x| > R \,.$$

We refer to [17, §3.2] for the complete set of assumptions which allow long range perturbations, at the expense of some analyticity assumptions standard for the definition of resonances – see [21] and references given there. We note that for $g^{ij} \equiv 1$ we obtain a class of semiclassical Schrödinger operators and for $V \equiv -1$, the Helmholtz equation for a Laplace-Beltrami operator, with $h = 1/\lambda$, playing the rôle of wavelength. The euclidean space can be without any changes replaced by a smooth manifold coinciding with $(\mathbb{R}^n \setminus B(0,R)) \sqcup \cdots \sqcup (\mathbb{R}^n \setminus B(0,R))$ outside of a compact set.

The key assumption is that the flow is hyperbolic on the trapped set at energy 0, K:

$$
\begin{gathered}
K \stackrel{\text{def}}{=} \{(x,\xi) \; : \; p(x,\xi)=0 \,, \quad \exp tH_p(x,\xi) \not\to \infty \,, \; t \to \pm\infty \} \,, \\
p(x,\xi) \stackrel{\text{def}}{=} \sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j + V(x) \,, \quad H_p(x,\xi) \stackrel{\text{def}}{=} \sum_{k=1}^{n} \partial_{\xi_k} p \, \partial_{x_k} - \partial_{x_k} p \, \partial_{\xi_k} \,, \\
\exp(tH_p)|_K \text{ is hyperbolic in the sense of [17, §3.3]}, \;\; dp|_{p^{-1}(0)} \neq 0 \,.
\end{gathered} \tag{1.2}
$$

The Hamiltonian flow defined by H_p corresponds to the geodesic flow on $S^*\mathbb{R}^n$ when $V = -1$, and the hyperbolicity assumption means that for $\rho \in K$, we have a flow invariant decomposition,

$$
T_\rho p^{-1}(0) = \mathbb{R}H_p(\rho) \oplus E_\rho^- \oplus E_\rho^+ \,,
$$

into neutral ($\mathbb{R}H_p(\rho)$), stable (E_ρ^-), and unstable (E_ρ^+) directions – see [17, (3.11)] for the full description.

The condition that the trapped set is "filamenentary" is formulated in terms of a topological pressure. We refer to [17, §3.3] and texts on dynamical systems [13],[25] for precise definitions, recalling only the definition in the special, but often satisfied, case in which the periodic orbits are dense in K. Let $f \in C^0(K)$. Then the *pressure* of f with respect to the Hamiltonian flow on K is given by

$$
\mathcal{P}(f) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T} \log \sum_{T_\gamma < T} \exp \int_0^{T_\gamma} (\exp tH_p)^* f \, dt \,, \tag{1.3}
$$

where the sum runs over all periodic orbits γ of periods $T_\gamma \leq T$.

We can now formulate our result:

Theorem. *Suppose that $P(h)$ satisfies* (1.1) *or the more general assumptions of* [17, §3.2]. *Suppose also that the Hamiltonian flow* (1.2) *is hyperbolic on the trapped set K, and that*

$$
\mathcal{P}(-\varphi_+/2) < 0 \,, \quad \varphi_+(\rho) \stackrel{\text{def}}{=} \frac{d}{dt} \left(d \exp tH_p|_{E^+\rho}(\rho) \right) |_{t=0} \,, \quad \rho \in K \,. \tag{1.4}
$$

Then for any $\chi \in \mathcal{C}_{\rm c}^\infty(\mathbb{R}^n)$ and $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ and $h(\epsilon) > 0$ such that the cut-off resolvent $\chi(P(h)-z)^{-1}\chi$, $\operatorname{Im} z > 0$, continues analytically to

$$
\Omega_\epsilon(h) \stackrel{\text{def}}{=} \{ z \; : \; \operatorname{Im} z > h(\mathcal{P}(-\varphi_+/2) + \epsilon) \,, \quad |\operatorname{Re} z| < \delta(\epsilon) \} \,, \quad 0 < h < h(\epsilon) \,.
$$

For $z \in \Omega_\epsilon(h) \cap \{\operatorname{Im} z \leq 0\}$, this resolvent is polynomially bounded in h:

$$
\begin{gathered}
\| \chi(P(h)-z)^{-1}\chi \|_{L^2\to L^2} \leq C(\epsilon,\chi) \, h^{-1+c_E \operatorname{Im} z/h} \log(1/h) \,, \\
c_E \stackrel{\text{def}}{=} \frac{n}{2|P(-\varphi_+/2) + \epsilon/2|} \,.
\end{gathered} \tag{1.5}
$$

We used the continuity of the pressure with respect to the energy in order to simplify the statement – see [17, Theorem 3] for a slightly more precise formulation. In dimension

$n = 2$ the condition (1.4) is equivalent to the statement that the Hausdorff dimension of $K \subset p^{-1}(0)$ is less than 2. Since the energy surface $p^{-1}(0)$ has dimension 3 and the minimal dimension of a non-empty K is 1, the condition means that we are less than "half-way" and K is *filamentary.* Trapped sets with dimensions greater than 2 are referred to as *bulky.*

The first part of the theorem is the main result of [17], see Theorem 3 there. Here we use the techniques developed in that paper to prove (1.5). For $-\Delta$ outside several convex obstacles (satisfying a condition guaranteeing strict hyperbolicity of the flow) with Dirichlet or Neumann boundary condition, the theorem was proved by Ikawa [12], with the pressure being only implicit in the statement which gave an explicit condition on distances and sizes of the obstacles. For more recent developments in that setting see [2],[18], and [15].

In particular, for z on the real axis the bound (1.5) gives

$$\|\chi(P(h)-z)^{-1}\chi\|_{L^2\to L^2} \leq C\,\frac{\log(1/h)}{h}\,, \quad z\in[-\delta(\epsilon),\delta(\epsilon)], \quad 0<h<h(\epsilon)\,. \tag{1.6}$$

This result was already given in [17, Theorem 5] with a less direct proof. It has been generalized to a larger class of manifolds in [7] and (1.5) provides no new insight in that setting. One of the applications is local smoothing with a minimal loss [5] (see [3] for the original application in the setting of obstacle scattering) and a recent no loss Strichartz estimate [4] under the assumption (1.4). The advantage of having (1.5) lies in applications to the energy decay for the wave equation – see [3, 6, 10] and references given there. That is particularly important in view of the recent results of Schenck on the use of pressure estimates in the setting of the damped wave equation [20].

To prove (1.5) we show in §3 that the estimates obtained in [17, §7] can be used to obtain a good parametrix for the complex-scaled operator, which leads to an estimate for the inverse. As was pointed out to us by Burq the construction of the parametrix for the outgoing resolvent was the, somewhat implicit, key step in the work of Ikawa [12] on the resonance gap for several convex obstacle. That insight lead us to re-examine the consequences of [17].

We follow the notation of [17] with precise references given as we go along. For the needed aspects of semiclassical microlocal analysis [17, §3] and the references to [8] and [9] should be consulted.

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2. Review of the hyperbolic dispersion estimates

The dispersion estimate we need is proved in [17] for a modified operator which we will now describe.

The first modification of $P(h)$ comes from the method of complex scaling reviewed in [17, §3.4]. For any fixed sufficiently large $R_0 > 0$, it results in the operator $P_\theta(h)$, with the following properties. To formulate them put

$$\Omega_\theta \stackrel{\text{def}}{=} [-\delta, \delta] + i[-\theta/C, C] \,, \quad \theta = M_1 h \log(1/h) \,. \tag{2.1}$$

Then

$$P_\theta(h) - z \;:\; H^2_h(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \;\text{ is a Fredholm operator for } z \in \Omega_\theta, \tag{2.2}$$

$$\forall\, \chi \in \mathcal{C}^\infty_{\rm c}(B(0, R_0)) \,, \quad \chi R(z,h)\chi = \chi R_\theta(z,h)\chi \,. \tag{2.3}$$

Here and below we set

$$R_\bullet(z,h) \stackrel{\text{def}}{=} (P_\bullet(h) - z)^{-1} \,, \quad \operatorname{Im} z > 0 \,,$$

and (2.3) shows the meromorphic continuation of $\chi R(z,h)\chi$ to Ω_θ, the meromorphy being guaranteed by the Fredholm property of $P_\theta(h) - z$.

The operator $P_\theta(h)$ is further modified by an exponential weight, $G^w = G^w(x, hD)$,

$$G \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n) \,, \quad \operatorname{supp} G \subset p^{-1}((-2\delta, 2\delta)) \,, \quad \partial^\alpha G = \mathcal{O}(h \log(1/h)) \,,$$

where $\delta > 0$ is a fixed number:

$$P_{\theta,\epsilon}(h) \stackrel{\text{def}}{=} e^{-\epsilon G^w/h} P_\theta(h) e^{\epsilon G^w/h} \,, \quad \epsilon = M_2\theta \,, \quad \theta = M_1 h \log(1/h) \,. \tag{2.4}$$

The new operator has the same spectrum as $P_\theta(h)$ and the following properties:

$$\operatorname{Im} \psi^w_0(x,hD)\, P_{\theta,\epsilon}(h)\, \psi^w_0(x,hD) \leq C\, h \tag{2.5}$$
$$\text{if } \operatorname{supp} \psi_0 \subset p^{-1}((-3\delta/2, 3\delta/2)) \,, \; \psi_0 \in S(1) \,,$$

$$\| \exp(\pm \epsilon G^w/h) \|_{L^2 \to L^2} = \mathcal{O}(h^{-CM_2}) \,. \tag{2.6}$$

The main reason for introducing the weight G is to ensure the bound (2.5). The specific choice of G is explained in [17, §6.1]. In particular G vanishes in some neibhbourhood of the trapped set K, and

$$\exp(tG^w(x,hD)) = B^w_t(x,hD) \,, \quad B_t \in h^{-N} S_\delta(T^*\mathbb{R}^n) \,, \quad B_t|_{\complement \operatorname{supp} G} = 1 + \mathcal{O}_{S_\delta}(h^\infty) \,.$$

Hence, if the spatial cutoff χ is supported away from $\pi \operatorname{supp} G$, calculus of semiclassical pseudodifferential operators ensures that

$$\chi R(z,h)\chi = \chi R_{\theta,\epsilon}(z,h)\chi + \mathcal{O}_{L^2\to L^2}(h^\infty) \| R_{\theta,\epsilon}(z,h) \| \,, \tag{2.7}$$

so our objective is to estimate $\| R_{\theta,\epsilon}(z,h) \|_{L^2 \to L^2}$.

We consider a final modification of $P(h)$ near the energy surface. Let $\psi_0 \in S(1)$ be supported in $p^{-1}((-3\delta/2, 3\delta/2)$ and equal to 1 in $p^{-1}(-\delta, \delta)$. Define

$$\widetilde{P}_{\theta,\epsilon}(h) \stackrel{\text{def}}{=} \psi^w_0(x,hD)\, P_{\theta,\epsilon}\, \psi^w_0(x,hD) \,, \tag{2.8}$$

and the modified propagator [17, (6.9)]:

$$U(t) \stackrel{\text{def}}{=} \exp\{-it\widetilde{P}_{\theta,\epsilon}(h)/h\} \,. \tag{2.9}$$

The crucial ingredients in proving (2.13) are the good upper bounds for the norms

$$\|U(t)\psi^w(x,hD)\|_{L^2\to L^2}\,,\quad 0\leq t\leq M\log(1/h)\,,$$

where $M>0$ is fixed but large, and

$$\psi\in S(1),\quad \operatorname{supp}\psi\subset p^{-1}((-\delta/2,\delta/2)),\quad \psi=1\ \text{ on }\ p^{-1}((-\delta/4,\delta/4))\,. \tag{2.10}$$

From the bound (2.5) on the imaginary part of the modified Hamiltonian, we get an exponential control on the propagator:

$$\|U(t)\|_{L^2\to L^2}\leq \exp(C\,t),\qquad t\geq 0\,. \tag{2.11}$$

The reason to conjugate P_θ with the weight G^w was indeed to ensure this exponential bound. Together with the hyperbolic bound (2.13), this exponential bound would suffice to get a polynomial bound $\mathcal{O}(h^{-L})$ in (1.5), for some (unknown) $L>0$. To obtain the explicit value,

$$-1+\frac{c_E\operatorname{Im}z}{h}\,,$$

for the exponent, we need to improve (2.11) into the following uniform bound:

Lemma 2.1. *Let ψ satisfy the conditions* (2.10)*. Then, there exists a constant $C_0>0$ such that, for any small enough h and any time $0\leq t\leq M\log(1/h)$,*

$$\|U(t)\psi^w(x,hD)\|_{L^2\to L^2}\leq C_0\,. \tag{2.12}$$

Before proving this Lemma, we state the major consequence of the *hyperbolic dispersion estimate* [17, Proposition 6.3], in the spirit of [17, §6.4] (see also [17, Proposition 9.1] for a simpler case). As above, we take ψ as in (2.10). For any $\epsilon>0$ and $0<h<h(\epsilon)$, we then have

$$\begin{gathered}\|U(t)\psi^w(x,hD)\|_{L^2\to L^2}\leq C\,h^{-n/2}\exp(-\lambda t)+\mathcal{O}(h^{M_3})\,,\\ \lambda\stackrel{\text{def}}{=}-\mathcal{P}(\varphi_+/2)+\epsilon/2\,,\end{gathered} \tag{2.13}$$

uniformly in the time range

$$0<t<M\log(1/h)\,.$$

We have used the notation of (1.4), M is arbitrarily large, and M_3 can be taken as large as we wish, provided we choose M_1 in (2.1) large enough depending on M.

If the pressure $\mathcal{P}(\varphi_+/2)$ is negative, one can take ϵ small enough to ensure $\lambda>\epsilon/2>0$. The above estimate is then sharper than (2.12) for times beyond the *Ehrenfest time*

$$t_E\stackrel{\text{def}}{=}c_E\,\log(1/h),\qquad c_E\stackrel{\text{def}}{=}\frac{n}{2\lambda}\,. \tag{2.14}$$

The large constant M will always be chosen (much) larger than c_E.

Proof of Lemma 2.12. To motivate the proof we start with a heuristic argument for the bound (2.12). As mentioned above, the exponential bound (2.11) is due to the fact that the imaginary part of the operator $\widetilde{P}_{\theta,\epsilon}(h)$ can take positive values of order $\mathcal{O}(h)$ (see (2.5)).

However, the construction of the weight G shows that outside a bounded region of phase space of the form

$$V_{\rm pos} = p^{-1}((-2\delta, 2\delta)) \cap T^*_{\{R_1<|x|<R_2\}}\mathbb{R}^n \,,$$

the imaginary part of $\widetilde{P}_{\theta,\epsilon}(h)$ is negative up to $\mathcal{O}(h^\infty)$ errors.

The radius R_1 above is large enough, so that $V_{\rm pos}$ lies *finite distance* from the trapped set. As a result, any trajectory crossing the region $V_{\rm pos}$ will only spend a bounded time in that region. For this reason, the propagator $U(t)$ on a large time $t \gg 1$ will "accumulate" exponential growth during a uniformly bounded time only.

We now provide a rigorous proof, using ideas and results from [17, §6.3]. The phase space $T^*\mathbb{R}^n$ is split using a smooth partition of unity:

$$1 = \sum_{b=0,1,2,\infty} \pi_b \,, \qquad \pi_b \in C^\infty(T^*\mathbb{R}^n, [0,1])\,.$$

These four functions have specific localization properties:

- $\operatorname{supp} \pi_b \subset p^{-1}((-\delta,\delta))$ for $b = 0, 1, 2$
- π_∞ is localized outside $p^{-1}((-3\delta/4, 3\delta/4))$
- π_1 is supported near K, in particular, its support does not intersect $V_{\rm pos}$
- π_2 is supported away from K but inside $\{|x| < R_2 + 1\}$
- π_0 is supported near spatial infinity, that is on $\{|x| > R_2 - 1\}$ where the operator $\widetilde{P}_{\theta,\epsilon}(h)$ is *absorbing* (the imaginary part of its symbol is negative).

Employing a positive (Wick) quantization scheme (see for instance [14], and for the semiclassical setting [19, §3.3]), $\Pi_b = \mathrm{Op}_h^+(\pi_b)$, we produce a quantum partition of unity

$$\mathrm{Id} = \sum_{b=0,1,2,\infty} \Pi_b \,, \qquad \|\Pi_b\| \leq 1\,.$$

It was the positive quantization that ensured $\|\Pi_b\| \leq 1$ above.

The evolution $U(t)$ is then split between time intervals of length t_0, where $t_0 > 0$ is large but independent of h. Using the partition of unity, we decompose the propagator at time $t = Nt_0$ into

$$U(Nt_0)\, \psi^w(x, hD) = \Big(\sum_{b=0,1,2,\infty} U_b \Big)^N \psi^w(x, hD), \quad \text{where} \quad U_b \stackrel{\rm def}{=} U(t_0)\, \Pi_b\,.$$

Expanding the power, we obtain a sum of terms $U_{b_N} \cdots U_{b_1} \psi^w$ and we will see that many of these terms are negligible.

Taking into account the energy cutoff $\psi \in C^\infty(p^{-1}((-\delta/2, \delta/2)))$ and assuming $N \leq C \log(1/h)$, it is shown in [17, Lemma 6.5] that any term containing at least one factor U_∞ (localized outside the energy shell) is $\mathcal{O}_{L^2\to L^2}(h^\infty)$.

Using the fact that any classical trajectory can travel in $\operatorname{supp}\pi_2$ at most for a finite time $\leq N_0 t_0$, [17, Lemma 6.6] shows that the *relevant* sequences $b_1 \cdots b_N$ are of the form

$$b_i = 1 \quad \text{for} \quad N_0 < i < N - N_0 \,.$$

They correspond to trajectories spending most of the time near K. Then [17, Lemma 6.5] shows that

$$U(Nt_0)\,\psi^w(x,hD) = U(N_0 t_0)\,(U_1)^{N-2N_0}\,U(N_0 t_0)\,\psi^w(x,hD) + \mathcal{O}_{L^2\to L^2}(h^{M_5})\,,$$

uniformly for any $2N_0 \leq N < M\log(1/h)$, where $M_5 > 0$.

Finally, [17, Lemma 6.3] shows that

$$U_1 = U(t_0)\Pi_1 = U_0(t_0)\Pi_1 + \mathcal{O}_{L^2\to L^2}(h^\infty)\,,$$

where $U_0(t_0) = \exp(-it_0 P(h)/h)$ is *unitary*. Hence, $\|U_1\| \leq 1 + \mathcal{O}(h^\infty)$, while $\|U(N_0 t_0)\|$ is estimated using (2.11). □

3. Resolvent estimates

We can now prove the resolvent estimate (1.5) by constructing a parametrix for $P_{\theta,\epsilon}(h)-z$, $z \in \Omega_\epsilon(h)$ defined in the statement of the theorem. We will use the notation

$$\zeta \stackrel{\text{def}}{=} z/h$$

to shorten some of the formulæ. We want to find an approximate solution to

$$(P_{\theta,\epsilon}(h) - z)u = f\,, \quad f \in L^2(\mathbb{R}^n)\,, \quad z \in \Omega_\epsilon(h)\,.$$

First, the ellipticity away from the energy surface $p^{-1}(0)$ shows that, for ψ as in (2.10), there exists an operator, $T_0 = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to H^2_h(\mathbb{R}^n)$, such that

$$(P_{\theta,\epsilon}(h) - z)T_0 f = (1 - \psi^w(x,hD))f + R_0 f\,, \qquad R_0 = \mathcal{O}_{L^2\to L^2}(h^\infty)\,.$$

To treat the vicinity of $p^{-1}(0)$ we put

$$T_1 f = (i/h)\int_0^{t_M} dt\, e^{i\zeta t}\, U(t)\,\psi^w(x,hD) f\,, \quad t_M = M\log(1/h)\,,$$

which satisfies

$$(\widetilde{P}_{\theta,\epsilon}(h) - z)\,T_1 f = \psi^w(x,hD)f + R_1 f\,, \qquad R_1 \stackrel{\text{def}}{=} -e^{i\zeta t_M}\,U(t_M)\,\psi^w(x,hD)\,. \tag{3.1}$$

The estimate (2.13) shows that, if $\lambda + \operatorname{Im}\zeta > \epsilon/2$, and for arbitrary $M_4 > 0$, one can choose M and M_3 large enough such that $R_1 = \mathcal{O}_{L^2\to L^2}(h^{M_4})$. We can estimate the norm of T_1 by the triangle inequality,

$$\|T_1\|_{L^2\to L^2} \leq h^{-1}\int_0^{t_M} e^{-\operatorname{Im}\zeta t}\,\|U(t)\,\psi^w(x,hD)\|_{L^2\to L^2}\,dt\,, \tag{3.2}$$

and then use the bounds (2.12) for times $0 \leq t \leq t_E$ and (2.13) for times $t_E < t \leq t_M$.

When $\operatorname{Im}\zeta = 0$, the above integral can be estimated by the integral over the interval $t \in [0, t_E]$:

$$\operatorname{Im}\zeta = 0 \Longrightarrow \|T_1\|_{L^2\to L^2} \leq h^{-1}\Big(C_0\, t_E + \frac{1}{\lambda}\Big) \leq C\, h^{-1}\, \log h^{-1}\,.$$

In the case $0 > \operatorname{Im}\zeta > -\lambda + \epsilon/2$, the dominant part of the integral comes from $t = t_E$:

$$0 > \operatorname{Im}\zeta > -\lambda + \epsilon/2 \Longrightarrow \|T_1\|_{L^2\to L^2} \leq C_\epsilon h^{-1}\, e^{-\operatorname{Im}\zeta t_E} = C_\epsilon\, h^{-1+c_E\operatorname{Im}\zeta}\,.$$

Now, (3.1) reads

$$\psi_0^w(x,hD)(P_{\theta,\epsilon}(h)-z)\psi_0^w(x,hD)T_1 f = \psi^w(x,hD)f + R_1 f\,.$$

From the inclusion $\psi_0|_{\operatorname{supp}\psi} \equiv 1$, one can show (as in [17, Lemma 6.5]) that

$$\psi_0^w(x,hD)(P_{\theta,\epsilon}(h)-z)\psi_0^w(x,hD)T_1 = (P_{\theta,\epsilon}(h)-z)\,T_1 + R_2\,, \qquad R_2 = \mathcal{O}_{L^2\to L^2}(h^\infty),$$

and also that

$$\|T_1\|_{H_h^2} \leq C\,\|T_1\|_{L^2}\,.$$

Putting $T = T_0 + T_1$ and $R = R_0 + R_1 + R_2$, we obtain

$$(P_{\theta,\epsilon}(h)-z)T = \operatorname{Id} + R\,, \qquad R = \mathcal{O}_{L^2\to L^2}(h^{M_4})\,.$$

This means that $(P_{\theta,\epsilon}(h)-z)$ can be inverted, with

$$\|(P_{\theta,\epsilon}(h)-z)^{-1}\|_{L^2\to H_h^2} = (1+\mathcal{O}(h^{M_4}))\|T\|_{L^2\to H_h^2}\,.$$

The above estimates on the norms of T_0 and T_1 can be summarized by

$$0 \geq \operatorname{Im}\zeta \geq \epsilon + \mathcal{P}(-\varphi_+/2) \Longrightarrow \|T\|_{L^2\to H_h^2} \leq C_\epsilon\, h^{-1+c_E\operatorname{Im}\zeta}\, \log h^{-1}\,. \tag{3.3}$$

Using (2.7), this proves the bound (1.5). □

Remark. By using a sharper energy cutoff ψ_h belonging to an exotic symbol class (see [23, §4]) and supported in the energy layer $p^{-1}((-h^{1-\delta}, h^{1-\delta}))$ (as in [1]), the bound (2.13) is likely to be improved to

$$\|U(t)\psi_h^w(x,hD)\| \leq C h^{-(n-1+\delta)/2}\exp(-\lambda t) + \mathcal{O}(h^{M_3})\,. \tag{3.4}$$

This bound becomes sharper than (2.12) around the time $t'_E = c'_E \log(1/h)$, where

$$c'_E \stackrel{\text{def}}{=} \frac{n-1+\delta}{2\lambda} < c_E.$$

As a result, the bounds on the norm of the corresponding operator T'_1 are modified accordingly. At the same time, as shown in [1, Prop. 5.4], the ellipticity away from the energy surface provides an operator T'_0 satisfying

$$(P_{\theta,\epsilon}(h)-z)T'_0 = (1-\psi_h^w(x,hD)) + \mathcal{O}_{L^2\to L^2}(h^\infty)\,,$$

and of norm $\|T'_0\|_{L^2\to H_h^2} = \mathcal{O}(h^{-1+\delta})$. The norm of $T' = T'_0 + T'_1$ is still dominated by that of T'_1, so that we eventually get

$$\|\chi(P(h)-z)^{-1}\chi\|_{L^2\to H_h^2} \leq C_\epsilon\, h^{-1+c'_E\operatorname{Im} z/h}\, \log(1/h)\,, \quad z \in \Omega_\epsilon(h) \cap \{\operatorname{Im} z \leq 0\}.$$

Since it is not clear that even this bound is optimal, and that proving (3.4) would require some effort, we have limited ourselves to using the established bound (2.13).

One advantage of the approach presented in this note (compared with the method of [17, §9]) is that, to obtain the bound (1.6) we did not have to use the complex interpolation arguments of [3] and [24].

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