# Soliton Home Movies

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MATHEMATIC
SCIENCES BUILDIN

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Apologies to my colleagues who may not quite agree...

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This equation has traveling wave solutions:

$$u(x,t) = e^{i\gamma(t)}\mu\mathrm{sech}\,(\mu(x-a-vt))\,,$$

$$\mu > 0\,,\quad v, a, \gamma \in \mathbf{R}\quad,$$

$$\gamma(t) = \gamma + vx + (\mu^2 - v^2)t/2\,.$$

$$\mu = 1\,, \quad v = 1\,, \quad a = -7\,.$$

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One of the amazing features in the stability of solitary waves in interaction. Collision of $\mu = 1$ and $\mu = 0.75$:

$$iu_t = -u_{xx}/2 - |u|^2 u\,, \quad u(x,0) = 2\mathrm{sech}x\,.$$

$$u(x,t) = 2e^{it/2}\mathrm{sech}x\left[(4 + 3\mathrm{sech}^2(e^{4it} - 1))/(4 - 3\mathrm{sech}^4 x \sin^2 2t)\right]$$

This solution is obtained using the inverse scattering method.

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Here $\delta_0$ is the famous Dirac delta function:

$$\delta_0(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} , \quad \int_{-\infty}^{\infty} \delta_0(x)dx = 1 .$$

$$q = 3\,, \quad v = 3\,, \quad x_0 = -3\,.$$

$$q = -0.02\,, \quad v_0 = 0\,, \quad a_0 = -3\,.$$

$$V(x) = -\operatorname{sech}^2(x/5)\,, \quad u_0(x) = \operatorname{sech}(x+3)\,.$$

$$V(x) = -\mathrm{sech}^2((x+5)/4) - \mathrm{sech}^2((x-5)/4) - 0.1\mathrm{sech}^2(x/4)\,,$$

$$u_0(x) = e^{ix/10}\mathrm{sech}(x+8)\,.$$

# Conclusions

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- The NLS models many phenomena such as the Bose-Einstein condensate, fiberoptics, impurities in DNA...
- Many phenomena hard to see numerically can be explained analytically
- And vice versa, many things easy to see numerically are hard analytically
- There are many open problems: long time behaviour, radiation and "breathing" patterns, multiple solitons interacting with impurities...