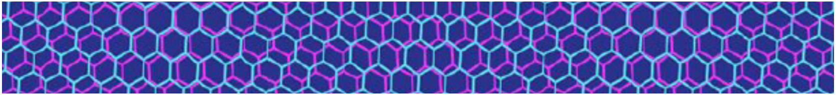
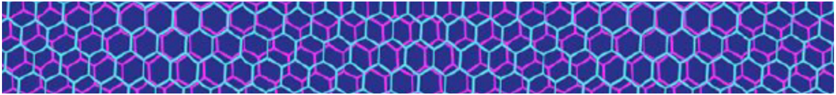
Mathematics of magic angles

Maxwell Institute Mini-symposium in Analysis and PDE

Maciej Zworski

March 31, 2023



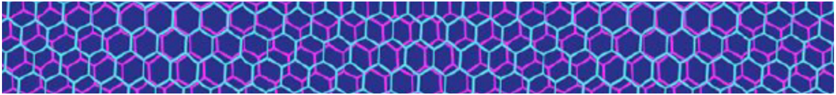


A project in the time of covid-19

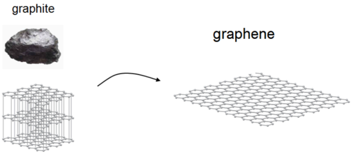
2020: Simon Becker, Mark Embree, Jens Wittsten, MZ: **BEWZ**

2022: Simon Becker, Tristan Humbert, MZ: **BHZ**

2023: Michael Hitrik, MZ: **HZ**, Simon Becker MZ: **BZ**



Motivation: bilayer graphene



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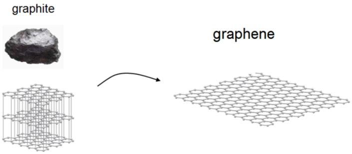

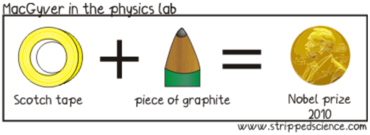

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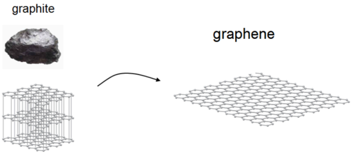

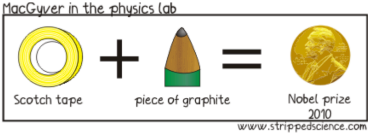

Geim–Novoselov '04

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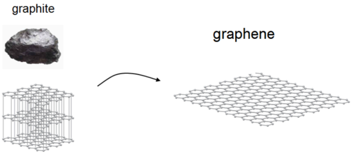

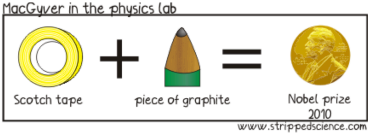

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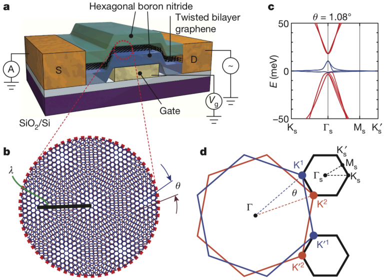

Cao et al '18, Yankovitz et al '18: superconductivity at $\theta \simeq 1.08^\circ$

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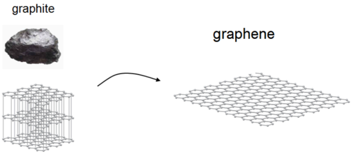

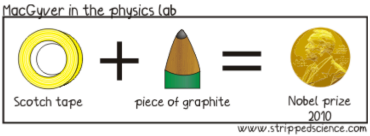

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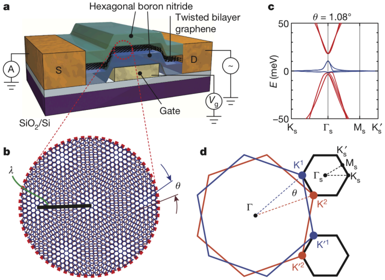

Cao et al '18, Yankovitz et al '18: superconductivity at $\theta \simeq 1.08°$

Predicted by Bistritzer–MacDonald '11

The chiral model of TBG

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PHYSICAL REVIEW LETTERS **122,** 106405 (2019)

Editors' Suggestion

Origin of Magic Angles in Twisted Bilayer Graphene

Grigory Tarnopolsky, Alex Jura Kruchkov,* and Ashvin Vishwanath
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The chiral model of TBG

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Derived from the full Bistritzer–MacDonald '11 Hamiltonian

The operator of today

$$D(\alpha) = \begin{pmatrix} 2D_{\bar z} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar z} \end{pmatrix} \quad \text{on } \mathbb{C}/\Gamma, \quad D_{\bar z} = \frac{1}{2i}(\partial_{x_1} + i\partial_{x_2})$$

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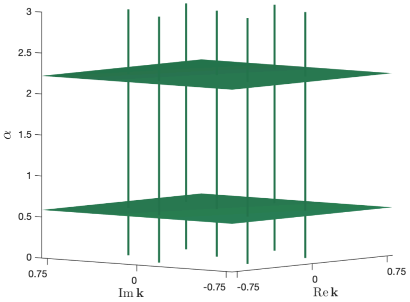

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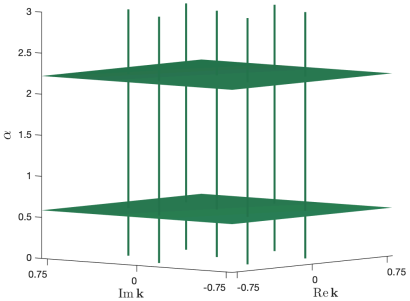

Seeley 85: $P(\alpha) = e^{ix} D_x + \alpha e^{ix}$, $x \in \mathbb{S}^1$, $\mathrm{Spec}(P(\alpha)) = \mathbb{C}$, $\alpha \in \mathbb{Z}$.

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Bands: eigenvalues of $H_{\mathrm{k}}(\alpha) := \begin{pmatrix} 0 & D(\alpha)^* - \bar{\mathrm{k}} \\ D(\alpha) - \mathrm{k} & 0 \end{pmatrix}$, $\mathrm{k} \in \mathbb{C}/\Gamma^*$

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A flat band at 0 energy means that $\mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) = \mathbb{C}$

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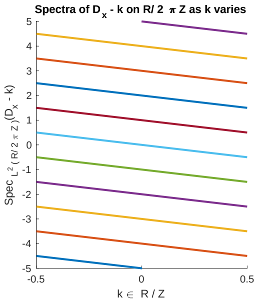
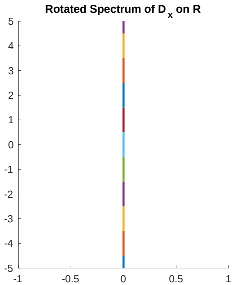
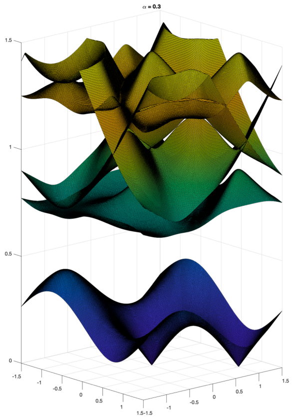




Flat bands

The bands are eigenvalues of $H_k(\alpha)$ on $L^2_0(\mathbb{C}/\Gamma)$, $k \in \mathbb{C}/3\Gamma^*$:



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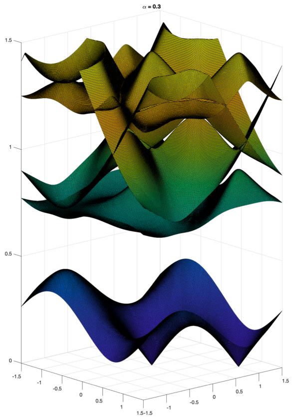

Theorem (BHZ '22; implicit in BEWZ '20)

$$\exists\, \mathrm{k} \notin 3\Gamma^* + \{0, -\mathrm{i}\} \;\; E_1(\alpha, \mathrm{k}) = 0 \implies \forall\, \mathrm{k} \;\; E_1(\alpha, \mathrm{k}) = 0.$$

A curious structure of the first band

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$$\mathsf{k} \mapsto \widetilde{E}_1(\alpha, \mathsf{k})/(\max_{\mathsf{k}} \widetilde{E}_1(\alpha, \mathsf{k})), \quad 0.4 < \alpha < 0.6$$

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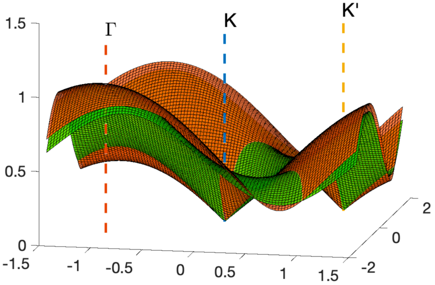

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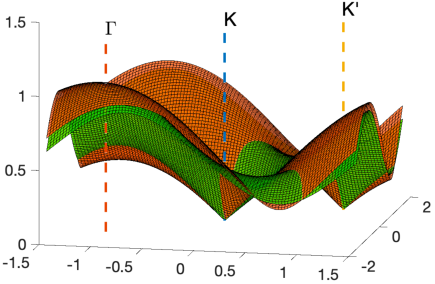

Rescaled plots remain almost fixed at $\mathrm{k} \longmapsto |U(-4\sqrt{3}\pi i \mathrm{k}/9)|$

Symmetries play a crucial role!

$$D(\alpha) = \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix}, \qquad H(\alpha) = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

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$$\mathscr{L}_{\mathsf{a}} \mathsf{u} = \mathrm{diag}(\omega^{a_1+a_2}, 1, \omega^{a_1+a_2}, 1) \mathsf{u}(z + \tfrac{4}{3} i\pi(\omega a_1 + \omega^2 a_2)), \quad \mathsf{a} \in \mathbb{Z}_3^2,$$

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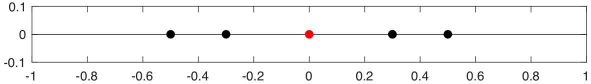

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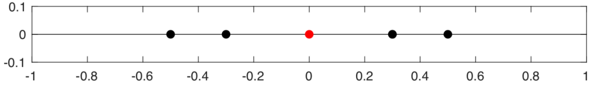

$$\dim \ker_{L^2(\mathbb{C}/\Gamma)}(H(\alpha)) \geq 4, \quad \dim \ker_{L^2(\mathbb{C}/\Gamma)}(D(\alpha)) \geq 2$$

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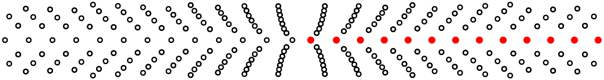

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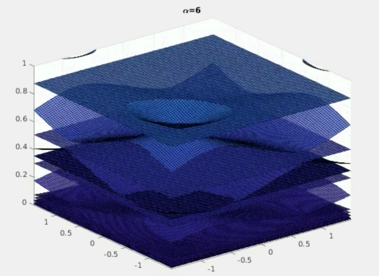

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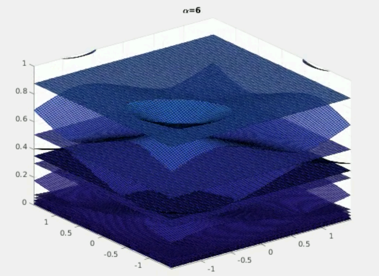

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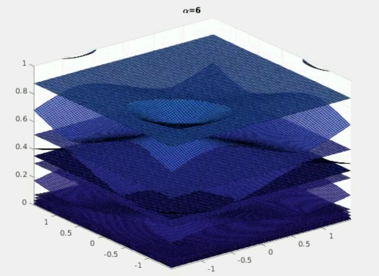

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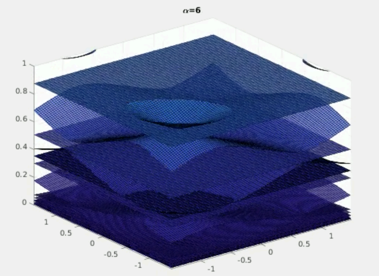

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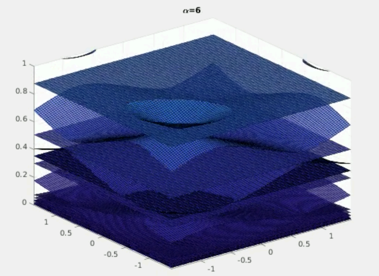

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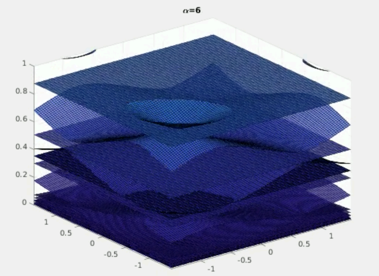

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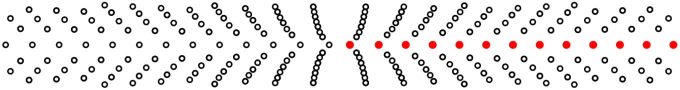

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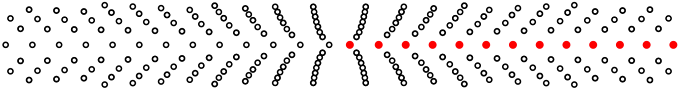

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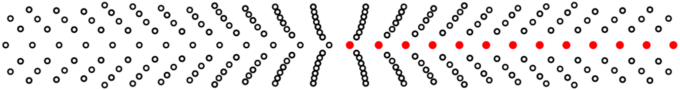

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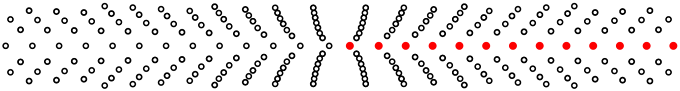

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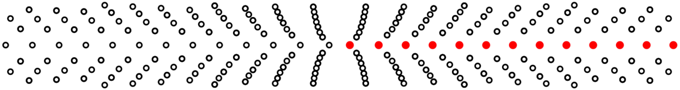

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p	$\sqrt{3}\sigma_p/3^p\pi$
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3	32/63
4	40/81

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2	4/9
3	32/63
4	40/81

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Theorem (BHZ '22) *The largest real eigenvalue of T_{k}, $1/\alpha_*$, is simple and $\alpha_* \in (0.583, 0.589)$.*

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k	α_k	$\alpha_k - \alpha_{k-1}$
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4	5.276497782985	1.5251
5	6.79478505720	1.5183
6	8.3129991933	1.5182
7	9.829066969	1.5161
8	11.34534068	1.5163
9	12.8606086	1.5153
10	14.376072	1.5155
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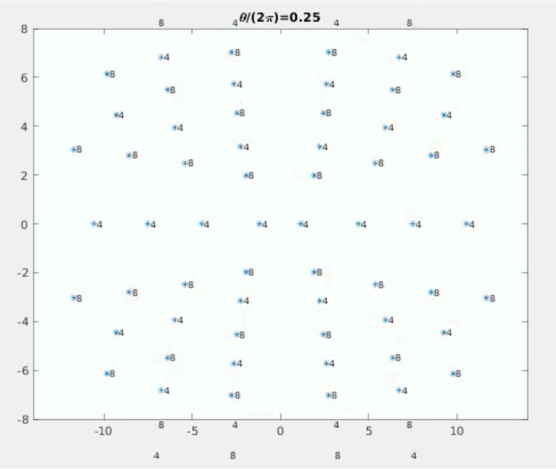

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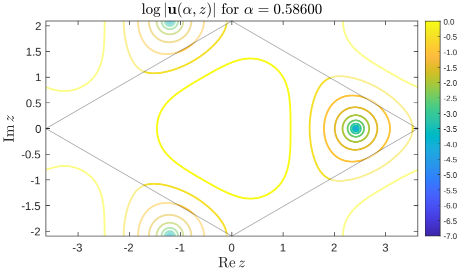

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Theorem (BHZ '22) $\alpha \in \mathcal{A}$ simple $\Rightarrow$ z_S *is the only zero of* u.

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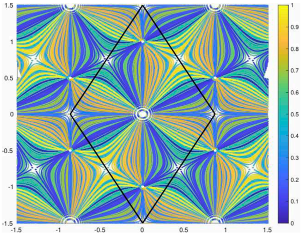

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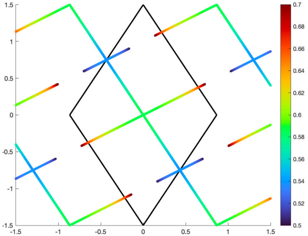

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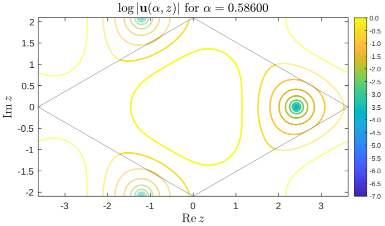

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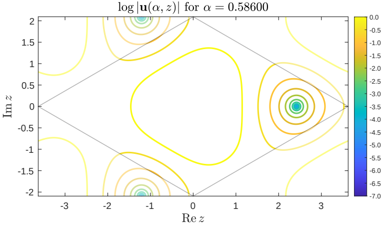

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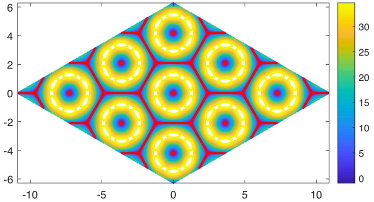

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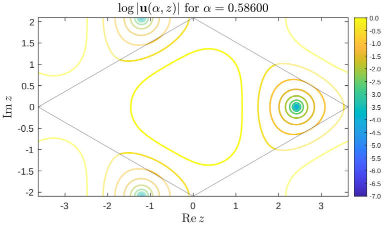

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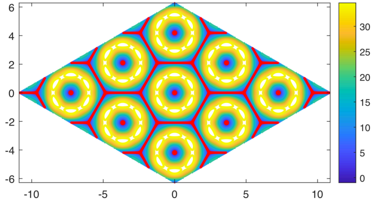

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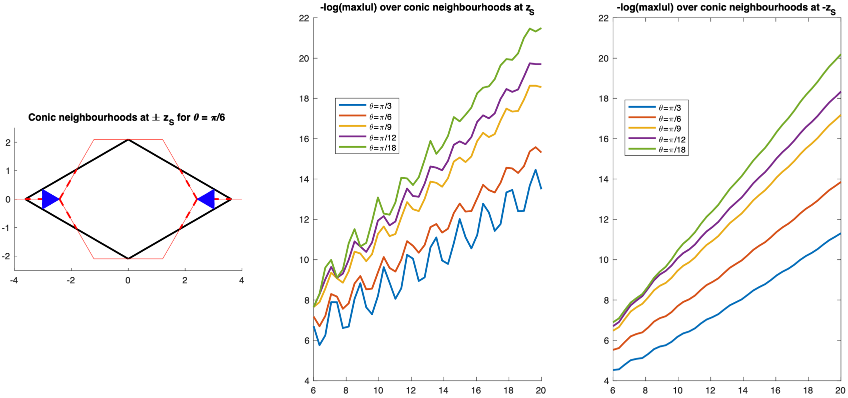

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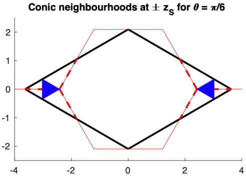

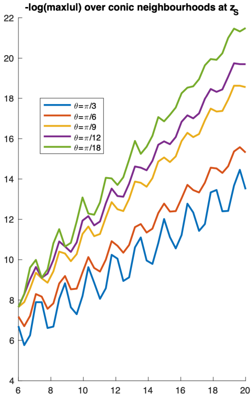

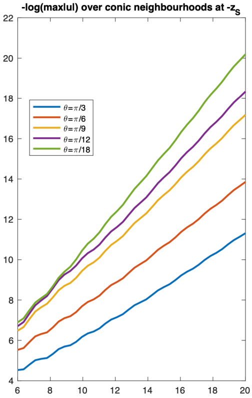

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$$\begin{pmatrix} 2hD_{\bar{z}} & U(z) \\ \alpha U(-z) & 2hD_{\bar{z}} \end{pmatrix} \mathrm{u} = 0 \implies ((2hD_{\bar{z}})^2 - U(z)U(-z) + hR)\mathrm{u} = 0$$

This allows an adaptation of (to some, v esoteric) hypoellipticity methods of Kashiwara, Sjöstrand, Trepreau, Himonas... (the 80's):

$$\{q, \bar{q}\}|_{\pi^{-1}(z_0)\cap q^{-1}(0)} = 0, \quad \{q, \{q, \bar{q}\}|_{\pi^{-1}(z_0)\cap q^{-1}(0)} \neq 0$$

implies the conclusion of the theorem for $\Omega = \mathrm{neigh}_{\mathbb{C}}(\mathrm{z}_0)$.

At the corners, it is trickier and does not fit into existing theories. Near the center of the hexagon q is not of principal type.

Another numerical observation (BHZ): Curvature

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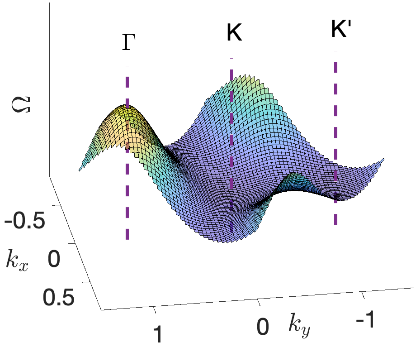

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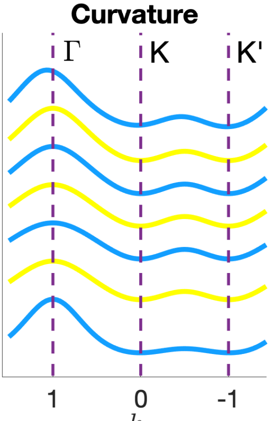

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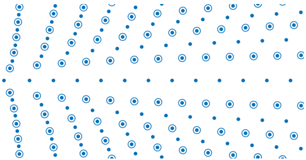

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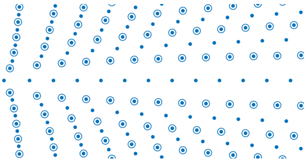

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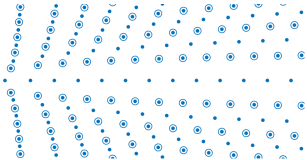

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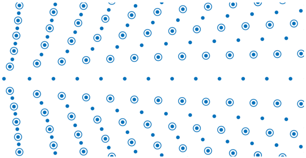

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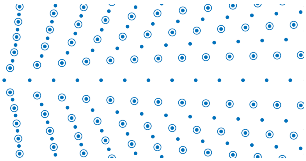

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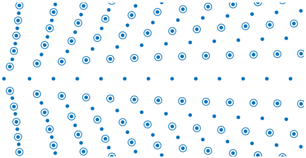

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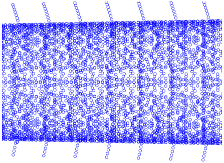

Thanks for your attention!