

Section 17.4

- (8)** Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$ and $-xy(x) = -\sum_{n=0}^{\infty} c_n x^{n+1} = -\sum_{n=1}^{\infty} c_{n-1} x^n$. The equation $y'' = xy$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0$ or $2c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - c_{n-1}]x^n = 0$. Equating coefficients gives $c_2 = 0$ and $c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \dots$. Since $c_2 = 0$, $c_{3n+2} = 0$ for $n = 0, 1, 2, \dots$. Given $c_0, c_3 = \frac{c_0}{3 \cdot 2}, c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \dots, c_{3n} = \frac{c_0}{3n(3n-1)(3n-3)(3n-4) \dots 6 \cdot 5 \cdot 3 \cdot 2}$. Given $c_1, c_4 = \frac{c_1}{4 \cdot 3}, c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}, \dots, c_{3n+1} = \frac{c_1}{(3n+1)3n(3n-2)(3n-3) \dots 7 \cdot 6 \cdot 4 \cdot 3}$. The solution can be written as
- $$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(3n-2)(3n-5) \dots 7 \cdot 4 \cdot 1}{(3n)!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(3n-1)(3n-4) \dots 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}$$

- (9)** Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum_{n=1}^{\infty} nc_n x^{n-1} = -\sum_{n=1}^{\infty} nc_n x^n = -\sum_{n=0}^{\infty} nc_n x^n$, $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$, and the equation $y'' - xy' - y = 0$ becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_n]x^n = 0$. Thus, the recursion relation is $c_{n+2} = \frac{nc_n + c_n}{(n+2)(n+1)} = \frac{c_n(n+1)}{(n+2)(n+1)} = \frac{c_n}{n+2}$ for $n = 0, 1, 2, \dots$. One of the given conditions is $y(0) = 1$. But $y(0) = \sum_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + \dots = c_0$, so $c_0 = 1$. Hence, $c_2 = \frac{c_0}{2} = \frac{1}{2}, c_4 = \frac{c_2}{4} = \frac{1}{2 \cdot 4}, c_6 = \frac{c_4}{6} = \frac{1}{2 \cdot 4 \cdot 6}, \dots, c_{2n} = \frac{1}{2^n n!}$. The other given condition is $y'(0) = 0$. But $y'(0) = \sum_{n=1}^{\infty} nc_n(0)^{n-1} = c_1 + 0 + 0 + \dots = c_1$, so $c_1 = 0$. By the recursion relation, $c_3 = \frac{c_1}{3} = 0, c_5 = 0, \dots, c_{2n+1} = 0$ for $n = 0, 1, 2, \dots$. Thus, the solution to the initial-value problem is
- $$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}$$

- (10)** Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2}$ and
- $$\begin{aligned} y''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-2}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2} \\ &= 2c_2 + 6c_3 x + \sum_{n=0}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2} \end{aligned}$$

Thus, the equation $y'' + x^2y = 0$ becomes $2c_2 + 6c_3x + \sum_{n=0}^{\infty} [(n+4)(n+3)c_{n+4} + c_n]x^{n+2} = 0$. So

$$c_2 = c_3 = 0 \text{ and the recursion relation is } c_{n+4} = -\frac{c_n}{(n+4)(n+3)}, n = 0, 1, 2, \dots$$

But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion relation, $c_{4n+1} = c_{4n+2} = c_{4n+3} = 0$ for $n = 0, 1, 2, \dots$

Also, $c_0 = y(0) = 1$, so

$$c_4 = -\frac{c_0}{4 \cdot 3} = -\frac{1}{4 \cdot 3}, c_8 = -\frac{c_4}{8 \cdot 7} = \frac{(-1)^2}{8 \cdot 7 \cdot 4 \cdot 3}, \dots, c_{4n} = \frac{(-1)^n}{4n(4n-1)(4n-4)(4n-5)\dots \cdot 4 \cdot 3}.$$

Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=0}^{\infty} c_{4n} x^{4n} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5)\dots \cdot 4 \cdot 3}$$

11. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $xy = x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$,

$$x^2 y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n+1},$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} \quad [\text{replace } n \text{ with } n+3]$$

$$= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1},$$

and the equation $y'' + x^2 y' + xy = 0$ becomes $2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + nc_n + c_n] x^{n+1} = 0$.

$$\text{So } c_2 = 0 \text{ and the recursion relation is } c_{n+3} = -\frac{nc_n - c_n}{(n+3)(n+2)} = -\frac{(n+1)c_n}{(n+3)(n+2)}, n = 0, 1, 2, \dots$$

But $c_0 = y(0) = 0 = c_2$ and by the recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \dots$

Also, $c_1 = y'(0) = 1$, so

$$c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{2}{4 \cdot 3}, c_7 = -\frac{5c_4}{7 \cdot 6} = (-1)^2 \frac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \frac{2^2 5^2}{7!}, \dots,$$

$$c_{3n+1} = (-1)^n \frac{2^2 5^2 \dots (3n-1)^2}{(3n+1)!}. \text{ Thus, the solution is}$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[(-1)^n \frac{2^2 5^2 \dots (3n-1)^2 x^{3n+1}}{(3n+1)!} \right]$$

12. (a) Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^{n+2}$,

$$xy'(x) = \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=-1}^{\infty} (n+2) c_{n+2} x^{n+2} = c_1 x + \sum_{n=0}^{\infty} (n+2) c_{n+2} x^{n+2}, \text{ and the equation}$$

$$x^2 y'' + xy' + x^2 y = 0 \text{ becomes } c_1 x + \sum_{n=0}^{\infty} \{[(n+2)(n+1) + (n+2)]c_{n+2} + c_n\} x^{n+2} = 0. \text{ So } c_1 = 0$$

$$\text{and the recursion relation is } c_{n+2} = -\frac{c_n}{(n+2)^2}, n = 0, 1, 2, \dots. \text{ But } c_1 = y'(0) = 0 \text{ so } c_{2n+1} = 0 \text{ for}$$

$$n = 0, 1, 2, \dots. \text{ Also, } c_0 = y(0) = 1, \text{ so } c_2 = -\frac{1}{2^2}, c_4 = -\frac{c_2}{4^2} = (-1)^2 \frac{1}{4^2 2^2} = (-1)^2 \frac{1}{2^4 (2!)^2},$$

$$c_6 = -\frac{c_4}{6^2} = (-1)^3 \frac{1}{2^6 (3!)^2}, \dots, c_{2n} = (-1)^n \frac{1}{2^{2n} (n!)^2}. \text{ The solution is}$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} (n!)^2}$$