

SECTION 17.4

1. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and the given equation, $y' - y = 0$, becomes

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0. \text{ Replacing } n \text{ by } n+1 \text{ in the first sum gives } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0,$$

$$\text{so } \sum_{n=0}^{\infty} [(n+1)c_{n+1} - c_n] x^n = 0. \text{ Equating coefficients gives } (n+1)c_{n+1} - c_n = 0, \text{ so the recursion relation is}$$

$$c_{n+1} = \frac{c_n}{n+1}, n = 0, 1, 2, \dots. \text{ Then } c_1 = c_0, c_2 = \frac{c_0}{2}, c_3 = \frac{1}{3} c_2 = \frac{1}{3} \cdot \frac{1}{2} c_0 = \frac{c_0}{3!}, c_4 = \frac{1}{4} c_3 = \frac{c_0}{4!}, \text{ and}$$

in general, $c_n = \frac{c_0}{n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x$$

2. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = xy \Rightarrow y' - xy = 0 \Rightarrow \sum_{n=1}^{\infty} n c_n x^{n-1} - x \sum_{n=0}^{\infty} c_n x^n = 0$ or

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0. \text{ Replacing } n \text{ with } n+1 \text{ in the first sum and } n \text{ with } n-1 \text{ in the second}$$

$$\text{gives } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0 \text{ or } c_1 + \sum_{n=1}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0. \text{ Thus,}$$

$$c_1 + \sum_{n=1}^{\infty} [(n+1)c_{n+1} - c_{n-1}] x^n = 0. \text{ Equating coefficients gives } c_1 = 0 \text{ and } (n+1)c_{n+1} - c_{n-1} = 0. \text{ Thus,}$$

$$\text{the recursion relation is } c_{n+1} = \frac{c_{n-1}}{n+1}, n = 1, 2, \dots. \text{ But } c_1 = 0, \text{ so } c_3 = 0 \text{ and } c_5 = 0 \text{ and in general } c_{2n+1} = 0.$$

$$\text{Also, } c_2 = \frac{c_0}{2}, c_4 = \frac{c_2}{4} = \frac{c_0}{4 \cdot 2} = \frac{c_0}{2^2 \cdot 2!}, c_6 = \frac{c_4}{6} = \frac{c_0}{6 \cdot 4 \cdot 2} = \frac{c_0}{2^3 \cdot 3!} \text{ and in general } c_{2n} = \frac{c_0}{2^n \cdot n!}. \text{ Thus, the}$$

solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{c_0}{2^n \cdot n!} x^{2n} = c_0 \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = c_0 e^{x^2/2}$$

3. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ and

$$-x^2 y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2} x^n. \text{ Hence, the equation } y' = x^2 y \text{ becomes}$$

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0 \text{ or } c_1 + 2c_2 x + \sum_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}] x^n = 0. \text{ Equating coefficients}$$

$$\text{gives } c_1 = c_2 = 0 \text{ and } c_{n+1} = \frac{c_{n-2}}{n+1} \text{ for } n = 2, 3, \dots. \text{ But } c_1 = 0, \text{ so } c_4 = 0 \text{ and } c_7 = 0 \text{ and in general}$$

$$c_{3n+1} = 0. \text{ Similarly } c_2 = 0 \text{ so } c_{3n+2} = 0. \text{ Finally } c_3 = \frac{c_0}{3}, c_6 = \frac{c_3}{6} = \frac{c_0}{6 \cdot 3} = \frac{c_0}{3^2 \cdot 2!},$$

$$c_9 = \frac{c_6}{9} = \frac{c_0}{9 \cdot 6 \cdot 3} = \frac{c_0}{3^3 \cdot 3!}, \dots, \text{ and } c_{3n} = \frac{c_0}{3^n \cdot n!}. \text{ Thus, the solution is}$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}$$

4. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$. Then the differential equation becomes $(x-3) \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$
 $\sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1} - 3 \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$
 $\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} [(n+2)c_n - 3(n+1)c_{n+1}] x^n = 0$
(since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the recursion relation is $c_{n+1} = \frac{(n+2)c_n}{3(n+1)}$, $n = 0, 1, 2, \dots$. Then $c_1 = \frac{2c_0}{3}$, $c_2 = \frac{3c_1}{3(2)} = \frac{3c_0}{3^2}$,
 $c_3 = \frac{4c_2}{3(3)} = \frac{4c_0}{3^3}$, $c_4 = \frac{5c_3}{3(4)} = \frac{5c_0}{3^4}$, and in general, $c_n = \frac{(n+1)c_0}{3^n}$. Thus the solution is
 $y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n$. [Note that $c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n = \frac{9c_0}{(3-x)^2}$ for $|x| < 3$.]

5. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation becomes $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$ or
 $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + nc_n + c_n] x^n$ (since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives
 $(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$, thus the recursion relation is $c_{n+2} = \frac{-(n+1)c_n}{(n+2)(n+1)} = -\frac{c_n}{n+2}$,
 $n = 0, 1, 2, \dots$. Then the even coefficients are given by $c_2 = -\frac{c_0}{2}$, $c_4 = -\frac{c_2}{4} = \frac{c_0}{2 \cdot 4}$, $c_6 = -\frac{c_4}{6} = -\frac{c_0}{2 \cdot 4 \cdot 6}$,
and in general, $c_{2n} = (-1)^n \frac{c_0}{2 \cdot 4 \cdots 2n} = \frac{(-1)^n c_0}{2^n n!}$. The odd coefficients are $c_3 = -\frac{c_1}{3}$, $c_5 = -\frac{c_3}{5} = \frac{c_1}{3 \cdot 5}$,
 $c_7 = -\frac{c_5}{7} = -\frac{c_1}{3 \cdot 5 \cdot 7}$, and in general, $c_{2n+1} = (-1)^n \frac{c_1}{3 \cdot 5 \cdot 7 \cdots (2n+1)} = \frac{(-2)^n n! c_1}{(2n+1)!}$. The solution is
 $y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$.