MATH 1B—SOLUTION SET FOR CHAPTERS 17.1 (#2), 17.2 (#1)

Problem 17.1.21. *Solve the initial-value problem*

$$\frac{y'' + 16y = 0, y(\pi/4) = -3, y'(\pi}{4) = 4}$$

Solution. The auxiliary equation is $r^2 + 16 = 0$, with solutions $r = \pm 4i$. Thus the general solution here is $y(x) = c_1 \sin 4x + c_2 \cos 4x$. Plugging in, we see $-3 = -c_2$, so $c_2 = 3$. Taking the derivative and plugging in, we see $4 = -4c_1$, or $c_1 = -1$. Thus the solution to the initial-value problem is $y(x) = 3\cos 4x - \sin 4x$.

Problem 17.1.24. *Solve the initial-value problem*

$$y'' + 12y' + 36y = 0, y(1) = 0, y'(1) = 1$$

Solution. The auxiliary equation here is $r^2 + 12r + 36 = 0$, which has a repeated solution at $r = -6$. The general form of the solution is therefore $y(x) = c_1 e^{-6x} + c_2 x e^{-6x}$. Now, at $x = 1$ we have $c_1 e^{-6} + c_2 e^{-6} = 0$, so $c_2 = -c_1$. Taking the derivative, we see $-6c_1 e^{-6} - c_1 e^{-6} + 6c_1 e^{-6} = 1$. Thus $c_1 = -e^6$, and $c_2 = e^6$. The solution to the initial-value problem is thus $y(x) = -e^{-6x+6} + xe^{-6x+6} = (x-1)e^{6-6x}$.

Problem 17.1.25. *Solve the boundary-value problem*

$$4y'' + y = 0, y(0) = 3, y(\pi) = -4$$

Solution. The auxiliary equation here is $4r^2 + 1 = 0$, so $r = \frac{1}{2}i$. The general form of the solution is thus $y(x) = c_1 \sin \frac{1}{2}x + c_2 \cos \frac{1}{2}x$. Now, $y(0) = 3$, so $c_2 = 3$. Moreover, $y(\pi) = c_1 = -4$. Thus, the solution to the boundary-value problem exists and is $y(x) = -4\sin \frac{1}{2}x + 3\cos \frac{1}{2}x$.

Problem 17.1.30. *Solve the boundary value problem*

$$y'' - 6y' + 9y = 0, y(0) = 1, y(1) = 0$$

Solution. The auxiliary equation of this differential equation is $r^2 - 6r + 9 = 0$, which has a repeated root at $r = 3$. The general solution is thus $y(x) = c_1 e^{3x} + c_2 x e^{3x}$. Now, we know $y(0) = 1$, so $c_1 = 1$. We also know that $y(1) = 0$, so $e^3 + c_2 e^3 = 0$. This implies $c_2 = -1$, so the solution to the boundary value problem is $y(x) = e^{3x} - xe^{3x}$ or $y(x) = (1-x)e^{3x}$.

Problem 17.1.33. *Let L be a nonzero real number.*

(a) The boundary-value problem $y'' + \lambda y = 0, y(0) = y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) For the case lambda > 0, there exist cases for which the problem has nontrivial solutions.

Solution.

(a) If $\lambda = 0$, we're simply considering the differential equation $y'' = 0$. Either by finding the roots of the auxiliary equation or by using common sense (we're looking for solutions with curvature zero, which are lines), we see that the general form of the solution is $y(x) = c_1 + c_2 x$. If $y(0) = 0$, then $c_1 = 0$; if $y(L) = 0$, then $c_2 L = 0$, whence $c_2 = 0$. In this case we have only the trivial solution.

If $\lambda < 0$, then (to avoid confusion) let $\gamma = -\lambda$, so $\gamma > 0$. Our differential equation is thus $y'' - \gamma y = 0$. The roots of the auxiliary equation are $\pm\sqrt{\gamma}$, so the general solution is $y(x) = c_1 e^{x\sqrt{\gamma}} + c_2 e^{-x\sqrt{\gamma}}$. Now, $y(0) = 0$, so $c_2 = -c_1$. Thus, the solution is of the form $y(x) = 2c_1 \sinh x\sqrt{\gamma}$. Now, we have $2c_1 \sinh L\sqrt{\gamma} = 0$, so (since sinh has only a single zero, at 0), $c_1 = 0$. Thus we again are limited to the trivial solution.

(b) If $\lambda > 0$, then the roots of the auxiliary equation are $\pm i\sqrt{\lambda}$. The general solution is thus $y(x) = c_1 \sin x\sqrt{\lambda} + c_2 \cos x\sqrt{\lambda}$. Now, $y(0) = 0$, so $c_2 = 0$. Since $y(L) = 0, c_1 \sin L\sqrt{\lambda} = 0$. If $c_1 = 0$, we're back at the trivial solution. There's another possibility this time, though: if $L\sqrt{\lambda} = n\pi$ for some integer n, then the sine function will be zero, regardless of the value of c_1. Thus, for

$$\lambda = \frac{\pi^2 n^2}{L^2}$$

the differential equation has nontrivial solutions.

Problem 17.2.1. *Solve using undetermined coefficients: $y'' + 3y' + 2y = x^2$.*

Solution. Well, the auxiliary equation for the complementary equation (As an aside, you have to love this terminology. Is it math, or just incoherent babbling? You decide!) is $r^2 + 3r + 2$, so we have roots $r \in \{-1, -2\}$. The homogeneous solution is thus $y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$. Now, the inhomogeneous part of the differential equation and all its derivatives are generated by the linearly independent set $\{x^2, x, 1\}$, so let's look for a particular solution of the form $y_p(x) = Ax^2 + Bx + C$. Plugging into the differential equation:

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = x^2$$

. Matching coefficients (this is why we need a linearly independent set!):

$$A = \frac{1}{2}, 6A + 2B = 0, 2A + 3B + 2C = 0$$

which gives us $A = \frac{1}{2}, B = -\frac{3}{2}, C = \frac{7}{4}$, so a particular solution is $y_p(x) = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$. Thus the solution to the differential equation is $y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$.

Problem 17.2.2. *Solve using undetermined coefficients:* $y'' + 9y = e^{3x}$.

Proof. The auxiliary equation is $r^2 + 9 = 0$, which has roots $\pm 3i$. The general solution to the homogeneous part of the differential equation is thus $y_h(x) = c_1 \sin 3x + c_2 \cos 3x$. Now, e^{3x} and all its derivatives are generated by e^{3x}, so let's look for a particular solution of the form Ae^{3x}. Plugging into the differential equation, we get $18Ae^{3x} = e^{3x}$, or $A = \frac{1}{18}$. Thus, $y_p(x) = \frac{1}{18}e^{3x}$, and the general solution is $y(x) = \frac{1}{18}e^{3x} + c_1 \sin 3x + c_2 \cos 3x$.

Problem 17.2.3. *Solve using undetermined coefficients:* $y'' - 2y' = \sin 4x$.

Solution. The auxiliary equation is $r^2 - 2r = 0$, with solutions $r \in \{0, 2\}$. The solution to the complementary equation is thus $y_h(x) = c_1 + c_2 e^{2x}$. Now, the driving term $\sin 4x$ and all its derivatives are generated by $\{\sin 4x, \cos 4x\}$. Thus, let's look for a particular solution of the form $y_p(x) = A \sin 4x + B \cos 4x$. Plugging into the differential equation, we see $-16A \sin 4x - 16B \cos 4x - 8A \cos 4x + 8B \sin 4x = \sin 4x$. Thus, matching coefficients we see that $-16A + 8B = 1, -16B - 8A = 0$. Thus $A = -2B$, so $40B = 1$, and $A = -\frac{1}{20}, B = \frac{1}{40}$. The particular solution is $y_p(x) = -\frac{1}{20}\sin 4x + \frac{1}{40}\cos 4x$, and the general solution $y(x) = -\frac{1}{20}\sin 4x + \frac{1}{40}\cos 4x + c_1 + c_2 e^{2x}$.

Problem 17.2.7. *Solve using the method of undetermined coefficients:* $y'' + y = e^x + x^3, y(0) = 2, y'(0) = 0$.

Solution. First, the auxiliary equation is $r^2 + 1 = 0$, with roots $\pm i$, so the homogeneous solution is $y_h(x) = c_1 \sin x + c_2 \cos x$. Now, the driving term is generated by $\{e^x, x^3, x^2, x, 1\}$, so we have to try a particular solution of the (unpleasant!) form $y_p(x) = Ae^x + Bx^3 + Cx^2 + Dx + E$. Plugging into the differential equation, we see

$$Ae^x + 6Bx + 2C + Ae^x + Bx^3 + Cx^2 + Dx + E = e^x + x^3.$$

Matching coefficients, we see:

$$2A = 1, B = 1, C = 0, 6B + D = 0, E = 0$$

This is readily solved to give particular solution $y_p(x) = \frac{1}{2}e^x + x^3 - 6x$, and so general solution $y(x) = c_1 \sin x + c_2 \cos x + \frac{1}{2}e^x + x^3 - 6x$. When $x = 0$, this is $y(0) = c_2 + \frac{1}{2} = 2$, so $c_2 = \frac{3}{2}$. The first derivative of the general solution is $y'(x) = c_1 \cos x - \frac{3}{2}\sin x + \frac{1}{2}e^x + 3x^2 - 6$. At $x = 0$, $y'(0) = c_1 + \frac{1}{2} - 6 = 0$, so $c_1 = \frac{11}{2}$. We thus have solution $y = \frac{11}{2}\sin x + \frac{3}{2}\cos x + \frac{1}{2}e^x + x^3 - 6x$.

Problem 17.2.10. *Solve using undetermined coefficients:*

$$y'' + y' - 2y = x + \sin 2x, y(0) = 1, y'(0) = 0$$

Solution. The auxiliary equation is $r^2 + r - 2 = 0$, which has roots $r = -2$ and $r = 1$. The homogeneous solution is thus $y_h(x) = c_1 e^x + c_2 e^{-2x}$. Now, the driving terms are generated by the set $\{x, 1, \sin 2x, \cos 2x\}$, and so we must try a particular solution of the form $y_p(x) = Ax + B + C \sin 2x + D \cos 2x$. Plugging into the differential equation, we get:

$$-4C \sin 2x - 4D \cos 2x + A + 2C \cos 2x - 2D \sin 2x - 2Ax - 2B - 2C \sin 2x - 2D \cos 2x = x + \sin 2x. \blacksquare$$

Matching coefficients, we see

$$-6C - 2D = 1, -6D + 2C = 0, -2A = 1, A - 2C = 0,$$

whence we get the particular solution $y_p(x) = -\frac{1}{2}x - \frac{1}{4} - \frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x$, and so the general solution $y(x) = c_1e^x + c_2e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x$.

Now, we must match initial conditions. Since

$$y'(x) = c_1e^x - 2c_2e^{-2x} - \frac{1}{2} - \frac{3}{10}\cos 2x + \frac{1}{10}\sin 2x,$$

plugging in conditions at $x = 0$ gives:

$$\begin{aligned} y(0) &= c_1 + c_2 - \frac{1}{4} - \frac{1}{20} = 1 \\ y'(0) &= c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} = 0 \end{aligned}$$

or

$$\begin{aligned} c_1 + c_2 &= \frac{13}{10} \\ c_1 - 2c_2 &= \frac{4}{5} \end{aligned}$$

So $c_2 = \frac{1}{6}, c_1 = \frac{17}{15}$, and we have our solution, $y(x) = \frac{17}{15}e^x + \frac{1}{6}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x$.