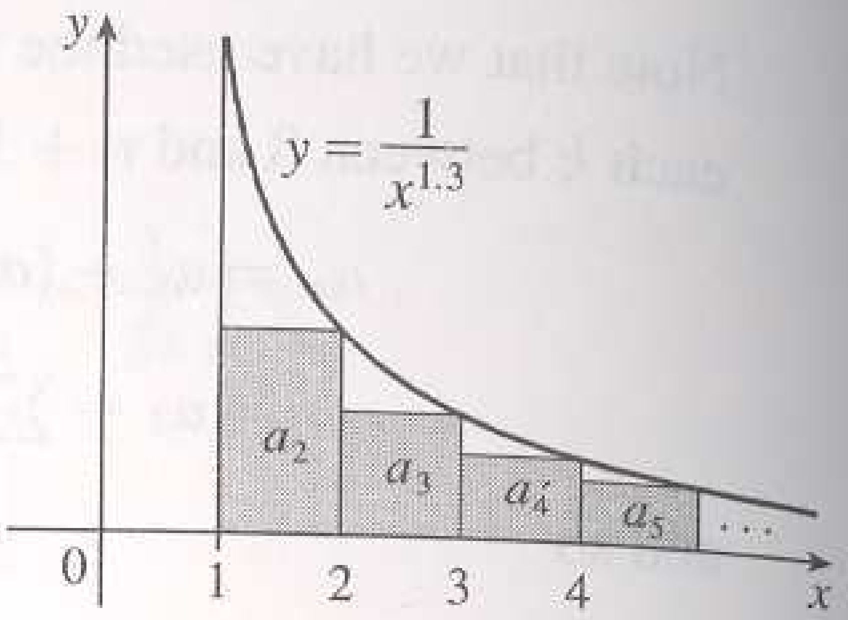
1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$,

$a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (7.8.2) with $p = 1.3 > 1$, so the series converges.



3. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_1^t x^{-4}\,dx = \lim_{t\to\infty} \left[\frac{x^{-3}}{-3}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{3t^3} + \frac{1}{3}\right) = \frac{1}{3}$. Since this improper integral is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent by the Integral Test.

5. The function $f(x) = 1/(3x+1)$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{dx}{3x+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty} \left[\tfrac{1}{3}\ln(3x+1)\right]_1^b = \lim_{b\to\infty} \left[\tfrac{1}{3}\ln(3b+1) - \tfrac{1}{3}\ln 4\right] = \infty$$

so the improper integral diverges, and so does the series $\sum_{n=1}^{\infty} 1/(3n+1)$.

8. The function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\displaystyle\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \left(1 + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} [x + \ln(x+1)]_1^t = \lim_{t\to\infty} (t + \ln(t+1) - 1 - \ln 2) = \infty$, so

$\displaystyle\int_1^\infty \frac{x+2}{x+1}\,dx$ is divergent and the series $\displaystyle\sum_{n=1}^\infty \frac{n+2}{n+1}$ is divergent. NOTE: $\displaystyle\lim_{n\to\infty} \frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

9. The series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ is a p-series with $p = 0.85 \le 1$, so it diverges by (1). Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ would have to converge (by Theorem 8(i) in Section 11.2).

16. The function $f(x) = \dfrac{3x+2}{x(x+1)} = \dfrac{2}{x} + \dfrac{1}{x+1}$ [by partial fractions] is continuous, positive, and decreasing on $[1, \infty)$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\int_1^\infty \frac{3x+2}{x(x+1)}\,dx = \lim_{t\to\infty} \int_1^t \left[\frac{2}{x} + \frac{1}{x+1}\right] dx = \lim_{t\to\infty} \left[2\ln x + \ln(x+1)\right]_1^t$$

$$= \lim_{t\to\infty} \left[2\ln t + \ln(t+1) - \ln 2\right] = \infty$$

33. $f(x) = x^{-3/2}$ is positive and continuous and $f'(x) = -\frac{3}{2}x^{-5/2}$ is negative for $x > 0$, so the Integral Test applies. From the end of Example 6, we see that the error is at most half the length of the interval. From (3), the interval is $\left(s_n + \int_{n+1}^{\infty} f(x)\,dx,\ s_n + \int_n^{\infty} f(x)\,dx\right)$, so its length is $\int_n^{\infty} f(x)\,dx - \int_{n+1}^{\infty} f(x)\,dx = \int_n^{n+1} f(x)\,dx$. Thus, we need n such that

$$0.01 > \frac{1}{2}\int_n^{n+1} x^{-3/2}\,dx = \frac{1}{2}\left[\frac{-2}{\sqrt{x}}\right]_n^{n+1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

$\Leftrightarrow\ n > 13.08$ (use a graphing calculator to solve $1/\sqrt{x} - 1/\sqrt{x+1} < 0.01$). Again from the end of Example 6, we approximate s by the midpoint of this interval. In general, the midpoint is $\frac{1}{2}\left[\left(s_n + \int_{n+1}^{\infty} f(x)\,dx\right) + \left(s_n + \int_n^{\infty} f(x)\,dx\right)\right] = s_n + \frac{1}{2}\left(\int_{n+1}^{\infty} f(x)\,dx + \int_n^{\infty} f(x)\,dx\right)$. So using $n = 14$, we have $s \approx s_{14} + \frac{1}{2}\left(\int_{14}^{\infty} x^{-3/2}\,dx + \int_{15}^{\infty} x^{-3/2}\,dx\right) \approx 2.0872 + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} \approx 2.6127 \approx 2.61$. Any larger value of n will also work. For instance, $s \approx s_{30} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{31}} \approx 2.6124$.

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

10. $\dfrac{n^2-1}{3n^4+1} < \dfrac{n^3}{3n^4+1} < \dfrac{n^3}{3n^4} = \dfrac{1}{3}\dfrac{1}{n}$. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series ($p = 2 > 1$). The terms of the given series are positive for $n > 1$, which is good enough.

11. If $a_n = \dfrac{n^2+1}{n^3-1}$ and $b_n = \dfrac{1}{n}$, then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+n}{n^3-1} = \lim_{n\to\infty} \frac{1+1/n^2}{1-1/n^3} = 1$, so $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$.

Or: Since $a_n = \dfrac{n^2+1}{n^3-1} > \dfrac{n^2+1}{n^3} > \dfrac{n^2}{n^3} = \dfrac{1}{n} = b_n$, we could use the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \dfrac{1+2^n}{1+3^n}$ and $b_n = \dfrac{2^n}{3^n}$: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{(1/2)^n + 1}{(1/3)^n + 1} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} < 1$), $\sum\limits_{n=1}^{\infty} \dfrac{1+2^n}{1+3^n}$ also converges.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}$ and $b_n = \dfrac{1}{3^n}$.

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + 7n}{n^2 + 5n - 1} = 2 > 0$, and since $\displaystyle\sum_{n=1}^{\infty} b_n$ is a convergent geometric series ($|r| = \frac{1}{3} < 1$),

$\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}$ converges also.

39. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N \Rightarrow 0 \leq a_n < 1$ for all $n > N \Rightarrow 0 \leq a_n^2 \leq a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 11.2.6, and $\sum b_n = \sum \sin(a_n)$ is a series with positive terms (for large enough n). We have $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = \lim\limits_{n\to\infty} \dfrac{\sin(a_n)}{a_n} = 1 > 0$ by Theorem 3.4.2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

2. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$. Here $a_n=(-1)^n\frac{n}{n+2}$. Since $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n+6}$. Now $b_n=\frac{4}{n+6}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty}b_n=0$, so the series converges by the Alternating Series Test.

4. $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$. $b_n=\frac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim_{n\to\infty}\frac{1}{\ln n}=0$, so the series converges by the Alternating Series Test.

5. $b_n=\frac{1}{\sqrt{n}}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty}b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n=\frac{1}{3n-1}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty}b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

14. $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$,

then $f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

23. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and

(ii) $\lim_{n\to\infty}\frac{1}{n^2} = 0$, so the series is convergent. Now $b_{10} = \frac{1}{10^2} = 0.01$ and $b_{11} = \frac{1}{11^2} = \frac{1}{121} \approx 0.008 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 10$. (That is, since the 11th term is less than the desired error, we need to add the first 10 terms to get the sum to the desired accuracy.)

25. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test

because $b_{n+1} = \dfrac{2^{n+1}}{(n+1)!} = \dfrac{2 \cdot 2^n}{(n+1)n!} = \dfrac{2}{n+1} \cdot \dfrac{2^n}{n!} = \dfrac{2}{n+1} \cdot b_n \le b_n$ and (ii)

$\displaystyle\lim_{n\to\infty} \frac{2^n}{n!} = \frac{2}{n} \cdot \frac{2}{n-1} \cdot \cdots \cdot \frac{2}{2} \cdot \frac{2}{1} = 0$, so the series is convergent. Now $b_7 = 2^7/7! \approx 0.025 > 0.01$ and $b_8 = 2^8/8! \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{4^n}$ satisfies (i) of the Alternating Series Test because

$b_{n+1} = \dfrac{n+1}{4^{n+1}} < \dfrac{n+3n}{4^n \cdot 4^1} = \dfrac{4n}{4 \cdot 4^n} = \dfrac{n}{4^n} = b_n$ and (ii) $\displaystyle\lim_{n\to\infty} \frac{n}{4^n} = 0$, so the series is convergent. Now $b_5 = 5/4^5 \approx 0.0049 > 0.002$ and $b_6 = 6/4^6 \approx 0.0015 < 0.002$, so by the Alternating Series Estimation Theorem, $n = 5$.

27. $b_7 = \dfrac{1}{7^5} = \dfrac{1}{16{,}807} \approx 0.000\,059\,5$, so

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx s_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^5} = 1 - \tfrac{1}{32} + \tfrac{1}{243} - \tfrac{1}{1024} + \tfrac{1}{3125} - \tfrac{1}{7776} \approx 0.972\,080$. Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.9721.