§11.8

3-20 Find the radius of convergence and interval of convergence of the series.

$$3. \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

We will apply the ratio test.

$$\left| \frac{x^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{x^n} \right| = \left| \frac{x\sqrt{n}}{\sqrt{n+1}} \right| \to |x| \qquad \text{as } n \to \infty.$$

Hence the radius of convergence is 1. For $x = 1$, the series is a divergent p-series, and for $x = -1$, the series is an alternating series, and since $\frac{1}{\sqrt{n}}$ is decreasing and converges to zero, the series converges. The interval of convergence is therefore $[-1, 1)$.

$$5. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^3}$$

$$\left| \frac{x^{n+1}}{(n+1)^3} \frac{n^3}{x^n} \right| = \left| \frac{x n^3}{(n+1)^3} \right| \to |x| \qquad \text{as } n \to \infty$$

Hence the radius of convergence is 1. For $x = \pm 1$, the series converges absolutely and therefore converges. Therefore the interval of convergence is $[-1, 1]$.

$$8. \sum_{n=1}^{\infty} n^n x^n.$$

$$\left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| = \frac{x(n+1)^{n+1}}{n^n}$$

converges if and only if $x = 0$. Therefore the radius of convergence is 0 and the interval of convergence is $[0, 0]$.

$$9. \sum_{n=1}^{\infty} (-1)^n n 4^n x^n$$

$$\left| \frac{(n+1)4^{n+1} x^{n+1}}{n 4^n x^n} \right| = \left| \frac{(4(n+1)x}{n} \right| \to |4x| \qquad \text{as } n \to \infty.$$

Therefore the radius of convergence is $\frac{1}{4}$. At the end points, $x = \pm\frac{1}{4}$, the sequence $(-1)^n n 4^n x^n$ diverges, so its sum cannot converge. Therefore the interval of convergence is $(-\frac{1}{4}, \frac{1}{4})$.

$$13. \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$$

$$\left| \frac{x^{n+1}}{4^{n+1} \ln(n+1)} \frac{4^n \ln n}{x^n} \right| = \left| \frac{x \ln n}{4 \ln(n+1)} \right| \to \left| \frac{x}{4} \right| \qquad \text{as } n \to \infty.$$

Therefore the radius of convergence is 4. For $x = 4$, the sequence

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$$

satisfies the criteria for the alternating series test and hence converges. For $x = -4$. the sequence

$$\sum_{n=2}^{\infty} (-1)^n \frac{(-4)^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges because for $n \geq 2$, $\frac{1}{n} \leq \frac{1}{\ln n}$, and the harmonic series diverges. The interval of convergence is therefore $(-4, 4]$.

$$14. \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\left| \frac{x^{2n+2}}{(2n+2)!} \frac{(2n)!}{x^{2n}} \right| = \frac{x^2}{(2n+1)(2n+2)} \to 0 \qquad \text{as } n \to \infty.$$

Therefore the radius of convergence is infinity and the interval of convergence is $\mathbb{R}$.

$$15. \sum_{n=0}^{\infty} \sqrt{n}(x-1)^n$$

$$\left| \frac{\sqrt{n+1}(x-1)^{n+1}}{\sqrt{n}(x-1)^n} \right| = \left| \frac{\sqrt{n+1}(x-1)}{\sqrt{n}} \right| \to |x-1| \qquad \text{as } n \to \infty.$$

The series converes if $|x-1| < 1$, so the radius of convergence is 1. If $x = 0$ or if $x = 2$, the series diverges because $\sqrt{n}(x-1)^n$ does not converge to zero. Therefore the interval of convergence is $(0, 2)$.

$$20. \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n 3^n}$$

$$\left|\frac{(3x-2)^{n+1}}{(n+1)3^{n+1}}\frac{n3^n}{(3x-2)^n}\right| = \left|\frac{n(3x-2)}{3(n+1)}\right| \to \left|x-\frac{2}{3}\right|, \qquad \text{as } n \to \infty.$$

The series converges if $|x-2/3| < 1$, so the radius of convergence is 1. If $x = 5/3$, the series is equal to the harmonic series and hence diverges. If $x = -1/3$, the series is equal to the alternating harmonic series and therefore converges. The interval of convergence is then $[-1/3, 5/3)$.

29. If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, does it follow that the following series are convergent?

$$(a) \sum_{n=0}^{\infty} c_n(-2)^n$$

Yes. If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, then the radius of convergence for the power series $\sum_{n=0}^{\infty} c_n x^n$ is at least 4. Therefore the interval of convergence contains -2.

$$(b) \sum_{n=0}^{\infty} c_n(-4)^n$$

No. Consider the power series

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{4^n n}.$$

Then the series converges for $x = 4$, because in that case it is the alternating harmonic series, but the series diverges for $x = -4$, because in that case it is equal to the positive harmonic series.

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n.$$

$$\left|\frac{[(n+1)!]^k x^{n+1}}{(kn+k)!}\frac{(kn)!}{(n!)^k x^n}\right| = \left|\frac{(n+1)^k x}{(kn+1)\cdot \ldots \cdot (kn+k)}\right| \to \left|\frac{x}{k^k}\right| \qquad \text{as } n \to \infty.$$

The radius of convergence is therefore k^k.

33. The function J_1 definted by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

is called the Bessel function of order 1.

(a) Find its domain.

$$\left|\frac{x^{2n+3}}{(n+1)!(n+2)!2^{2n+3}}\frac{n!(n+1)!2^{2n+1}}{x^{2n+1}}\right| = \frac{x^2}{4(n+1)(n+2)} \to 0 \qquad \text{as } n \to \infty.$$

Therefore the domain of J_1 is $\mathbb{R}$.

34. The function A defined by

$$A(x) = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 9} + \cdots$$

is called the Airy function after the English mathematician and astronomer Sir George Airy.

(a) Find the domain of the Airy function.

If we write $A(x) = \sum_{n=0}^{\infty} a_n x^{3n}$, then we find that

$$a_n = \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot \ldots \cdot (3n-1)\cdot 3n}.$$

Since

$$\left|\frac{x^{3n+3}a_{n+1}}{x^{3n}a_n}\right| = \left|\frac{x^3}{(3n-1)\cdot 3n}\right| \to 0 \qquad \text{as } n \to \infty,$$

the series converges for all values of x in $\mathbb{R}$.