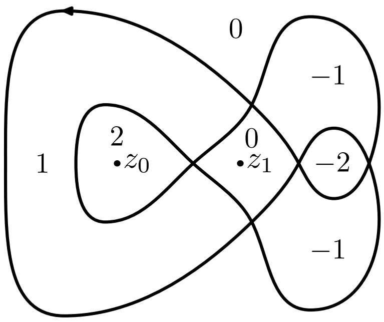
Solutions to the second midterm

Christian Zickert

Problem 1 (25 points). Let γ be a C^1 curve parametrizing the path shown below in the direction indicated by the arrow.



(a) **(10 points).** For each component C of $\mathbb{C} \setminus \gamma$, determine the winding number of γ about the points in C.

Solution: By a routine calculation, the winding numbers are as indicated.

(b) **(15 points).** Let z_0 and z_1 be the points indicated, and let f be the function given by

$$f(z) = \frac{\exp(1/(z-z_1))(z-z_0)^2}{\sin^2(z-z_0)}.$$

Show that $\int_\gamma f = 0$.

Solution: Clearly f is holomorphic in $\mathbb{C} \setminus \{z_0, z_1\}$. Since $\lim_{z\to 0} \frac{\sin(z)}{z} = 1$, it follows that f has a removable singularity at z_0. By the residue theorem,

$$\int_\gamma f = 2\pi i\big(\operatorname{ind}_\gamma(z_0)\operatorname{res}_{z_0}(f) + \operatorname{ind}_\gamma(z_1)\operatorname{res}_{z_1}(f)\big) = 2\cdot 0 + 0\cdot \operatorname{res}_{z_0}(f) = 0.$$

Problem 2 (30 points). Locate the isolated singularities (in $\mathbb{C}$) of the functions given below. Determine the type of each singularity (removable, pole of order k, or essential) and compute its residue.

(a) **(6 points).** $f(z) = \frac{\sin(z)\cos(z)}{z^2}$

Solution: The only singularity of f is at 0. Since

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \big(\frac{\sin(z)}{z}\cos(z)\big) = 1 \cdot \cos(0) = 1,$$

f has a simple pole at 0 with residue 1.

(b) **(8 points).** $f(z) = \cos(1/z) + \sin(1/z)$.

Solution: Again, 0 is the only singularity. We have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-2n-1} \\ &= \cdots - \frac{z^{-2}}{2} + z^{-1} + 1, \end{aligned}$$

from which it follows that 0 is an essential singularity with residue 1.

(c) **(8 points).** $f(z) = \frac{2z+3}{z^2(z+1)}$

Solution: The singularities are at 0 and -1. Since

$$\lim_{z\to -1} (z+1)f(z) = \lim_{z\to -1} \frac{2z+3}{z^2} = 1,$$

it follows that -1 is a simple pole with residue 1. To find the type of the singularity at 0, observe that

$$\begin{aligned} f(z) = (\frac{3}{z^2} + \frac{2}{z})\frac{1}{1+z} &= (\frac{3}{z^2} + \frac{2}{z})\sum_{n=0}^{\infty}(-z)^n \\ &= (\frac{3}{z^2} + \frac{2}{z}) - z(\frac{3}{z^2} + \frac{2}{z}) + \ldots \\ &= \frac{3}{z^2} - \frac{1}{z} + \ldots. \end{aligned}$$

It follows that 0 is a double pole (pole of order 2) with residue -1.

(d) **(8 points).** $f(z) = \frac{\log(z+1)}{z}$, where $|z| < 1$ and log is the unique branch of logarithm with $\log(1) = 0$.

Solution: The only singularity is 0, and since $\log(1) = 0$, we have $\log(1+z) = zg(z)$, where g is a holomorphic function. It follows that the limit $\lim_{z\to 0} zf(z)$ exists (and equals $g(0)$). Hence, 0 is a removable singularity, so the residue is 0.

Problem 3 (20 points). Let $f(z) = \frac{e^z}{z}$, and let γ be a counterclockwise parametrization of the unit circle.

(a) **(10 points).** Compute $\int_\gamma f$.

Solution: The only singularity of f is 0, and one easily checks that the residue is 1. Hence, by the residue theorem $\int_\gamma f = 2\pi i$

(b) **(10 points).** Use your result above to compute the integrals $\int_0^{2\pi} e^{\cos(t)} \cos(\sin(t))dt$ and $\int_0^{2\pi} e^{\cos(t)} \sin(\sin(t))dt$.

Solution: Let $\gamma(t) = e^{it}$. By the above, we have

$$\begin{aligned} 2\pi i &= \int_\gamma \frac{e^z}{z} dz \\ &= \int_0^{2\pi} \frac{e^{e^{it}}}{e^{it}} i e^{it} dt \\ &= i \int_0^{2\pi} e^{\cos(t)+i\sin(t)} dt \\ &= i \int_o^{2\pi} e^{\cos(t)} \big(\cos(\sin(t)) + i \sin(\sin(t))\big) dt. \end{aligned}$$

Equating real and imaginary parts, we obtain

$$\int_0^{2\pi} e^{\cos(t)} \cos(\sin(t))dt = 2\pi, \qquad \int_0^{2\pi} e^{\cos(t)} \sin(\sin(t))dt = 0.$$

Problem 4 (25 points).

(a) **(12 points).** Show that the polynomial $z^4 + 2z - 5$ has no zeros with norm less than or equal to 1.
Hint: Rouché's theorem.

Solution: We have $|z^4+2z-5| \geq \big||z^4+2z|-5\big|$, and since $1 \leq |z^4+2z| \leq 3$ when $|z| = 1$, we have

$$|f(z) - (2z-5)| = |z^4| = 1 < |f(z)|, \quad \text{when } |z| = 1.$$

In particular, $f(z)$ has no zeros with norm 1. Since the only zero of $g(z) = 2z - 5$ has norm $5/2 > 1$, it follows from Rouché's theorem, that $f(z)$ has no zeros with $|z| < 1$.

(b) **(13 points).** Compute the integral $\int_\gamma \frac{\sin(\pi z)}{(2z-1)(z^4+2z-5)} dz$, where γ is a counterclockwise parametrization of the unit circle.

Solution: By the above, $\frac{\sin(\pi z)}{(2z-1)(z^4+2z-5)}$ has no poles on γ and only one pole at $1/2$ in the interior of γ. The residue at $1/2$ equals $\frac{\sin(\pi/2)}{2((1/2)^4+2(1/2)-5)} = -8/63$. Hence, by the residue theorem, we have

$$\int_\gamma \frac{\sin(\pi z)}{(2z-1)(z^4+2z-5)} dz = -16\pi i/63.$$