HW 4 Solutions

Hey guys. This is your reader. Sorry I've been behind...it's a combination of sickness and grad school visitation. There were parts of the last two problems on HW 4 that need clarification, so I wrote up detailed solutions for them. If you'd like me to continue this service, please email me at `charleschen@berkeley.edu`. You can also talk to me about homework help, complaints and other crap.

Exercise 9.6

We want to find the image of the intersection of $I = \{|z| < 1\}$ (inside of the unit circle) and $H_+ = \{\Re(z) > 0\}$ (upper plane.) Since ϕ is bijective, we get $\phi(I \cap H_+) = \phi(I) \cap \phi(H_+)$. So let's find the image of I and H_+.

(WARNING: the thing I did with the intersections is not true in general! If we take $f(x) = x^2$, a real-valued function and let $A = (-\infty, 0]$, $B = [0, \infty)$, we get $f(A \cap B) = \{0\}$, but $f(A) \cap f(B) = \mathbb{R}$!)

First, you have $\phi(1) = 0$, $\phi(0) = -1$, and $\phi(-1) = \infty$. Since FLTs map clircles to clircles, and clircles are determined by three points, the extended real line gets sent to the extended real line.

The extended real line separates $\overline{\mathbb{C}}$ into two half planes: the top half plane H_+ and the bottom half plane H_-. Since ϕ is a bijection, and the real axis gets mapped to the real axis, $H_+ \cup H_-$ has to get mapped to $H_+ \cup H_-$. So

$$\phi(H_+ \cup H_-) = \phi(H_+) \cup \phi(H_-) = H_+ \cup H_-.$$

Since ϕ is continuous, it must send connected sets to connected sets. Hence, $\phi(H_+)$ and $\phi(H_-)$ are connected. $\phi(H_+)$ can't touch both H_+ and H_-, since it's connected. Hence, $\phi(H_+) \subset H_+$ or $\phi(H_+) \subset H_-$. Since $\phi(i) = i$, it must be the first possibility: $\phi(H_+) \subset H_+$.

Similarly, $\phi(H_-) \subset H_-$. Finally, since ϕ is a bijection, the inclusions can't be proper, so $\phi(H_+) = H_+$ and $\phi(H_-) = H_-$.

Next, where does the unit circle get mapped? $\phi(1) = 0$, $\phi(-1) = \infty$, and $\phi(i) = i$. Since clircles map to clircles, the unit circle must get mapped to the imaginary axis.

Let $I = \{|z| < 1\}$ (inner), $O = \{|z| > 1\}$ (outer), $L = \{\Im(z) < 0\}$ (left), and $R = \{\Im(z) > 0\}$ (right). Since ϕ is a bijection, we know

$$\phi(I) \cup \phi(O) = L \cup R.$$

Similar proof as before tells us that $\phi(I)$ is either L or R. We know earlier that $\phi(0) = -1$, so $\phi(I) = L$.

Hence, $\phi(I \cap H_+) = \phi(I) \cap \phi(H_+) = L \cap H_+$.

Exercise 9.7

Damn..$f(z)$ is not a LFT. But $g(z) = \frac{z+1}{z-1}$ is!

(a) $g(0) = -1$, $g(1) = \infty$ and $g(-1) = 0$. So g maps the extended real line to the extended real line. Squaring the reals gives nonnegative reals (and squaring ∞ is still ∞), so the image is the extended nonnegative real axis.

(b) $g(0) = -1$, $g(i) = i$, and $g(-i) = -i$. So g maps the extended imaginary line to the unit circle. Squaring the unit circle still gives you the unit circle, so the image is the unit circle.

(c) Because g sends the imaginary axis to the unit circle, a similar analysis from the previous problem shows that g must send R (the right half-plane) to either I (the inside of the unit circle) or O (the outside of the unit circle.) Since $g(1) = \infty$, $g(R)$ must equal O. Squaring O will still give you O. (why?)