Solutions to the first midterm

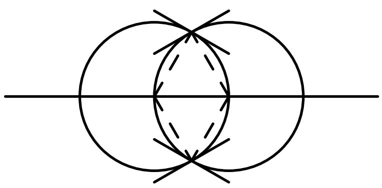
Christian Zickert

Problem 1 (20 points). Find the radius of convergence for the Taylor expansion of the function $f(z) = 3/(1 - 5iz)$ at the origin.

Solution: First note that f is holomorphic in $\mathbb{C} \setminus \{-i/5\}$. The Taylor expansion at the origin converges to f in any open disk, centered at the origin, in which f is holomorphic. The largest such disk has radius $1/5$. Hence, the radius of convergence is $1/5$.

Problem 2 (25 points). Let C_1 and C_2 be the circles in $\mathbb{C}$ of radius 1 centered at 0 and 1, respectively. Show that there can not exist a linear fractional transformation taking C_1 to $\mathbb{R} \cup \{\infty\}$ and C_2 to $i\mathbb{R} \cup \{\infty\}$.

Solution: From the figure below it follows that the tangents of the two circles make an angle of 120 degrees at both the points of intersection. Since the real and imaginary axis make an angle of 90 degrees, it follows from conformality that no such function exists.



Problem 3 (25 points). Let U be the open subset of $\mathbb{C}$ consisting of the complex numbers z for which $|z| < 1$. Let $f\colon U \to \mathbb{C}$ be given by $f(z) = -2/(1 - z)$. Show that f has a logarithm in U, i.e. show that there exists a holomorphic function $g\colon U \to \mathbb{C}$ such that $\exp(g(z)) = f(z)$ for all $z \in U$.

Solution: A simple computation shows that f satisfies

$$f(\pm i) = -1 \mp i, \quad f(1) = \infty, \quad f(0) = -2.$$

Since f is a linear fractional transformation, f preserves circles in $\mathbb{C} \cup \{\infty\}$, and it follows that f must map U to the set $V = \{w \in \mathbb{C} \mid \mathrm{Re}(w) < -1\}$.

Let $\log_0$ denote the branch of logarithm with argument in $[0, 2\pi)$. Then $\log_0$ is holomorphic in V, and we can set $g = \log_0 \circ f$.

Problem 4 (30 points). For $r > 0$ let $\gamma_r \colon [0, \pi] \to \mathbb{C}$ be the half circle given by $\gamma(t) = re^{it}$.

(a) **(15 points).** Show that $\left| \int_{\gamma_r} z^{-1} e^{iz} dz \right| \leq \int_0^\pi e^{-r\sin(t)} dt$.

Solution: We have

$$\begin{aligned} \left| \int_{\gamma_r} z^{-1} e^{iz} dz \right| &= \left| \int_0^\pi \gamma_r(t)^{-1} e^{i\gamma_r(t)} \gamma_r'(t) dt \right| \\ &= \left| \int_0^\pi \frac{1}{re^{it}} e^{ire^{it}} ire^{it} dt \right| \\ &\leq \int_0^\pi |e^{ir(\cos(t) + i\sin(t))}| dt \\ &= \int_0^\pi e^{-r\sin(t)} dt. \end{aligned}$$

(b) **(15 points).** Show that $\int_{\gamma_r} z^{-1} e^{iz} dz \to 0$ when $r \to \infty$.

Hint: $\int_0^\pi e^{-r\sin(t)} dt = \int_0^\varepsilon e^{-r\sin(t)} dt + \int_\varepsilon^{\pi-\varepsilon} e^{-r\sin(t)} dt + \int_{\pi-\varepsilon}^\pi e^{-r\sin(t)} dt$.

Solution: Let ε be a small positive number. We have

$$\begin{aligned} \int_0^\pi e^{-r\sin(t)} dt &= \int_0^\varepsilon e^{-r\sin(t)} dt + \int_\varepsilon^{\pi-\varepsilon} e^{-r\sin(t)} dt + \int_{\pi-\varepsilon}^\pi e^{-r\sin(t)} dt \\ &= 2\int_0^\varepsilon e^{-r\sin(t)} dt + \int_\varepsilon^{\pi-\varepsilon} e^{-r\sin(t)} dt. \end{aligned}$$

Note that $\sin(t) \geq t/2$ for $0 \leq t \leq \varepsilon$, and that $\sin(t) \geq \sin(\varepsilon)$ for $\varepsilon \leq t \leq \pi - \varepsilon$. It now follows that

$$\begin{aligned} \int_0^\pi e^{-r\sin(t)} dt &\leq 2\int_0^\varepsilon e^{-rt/2} dt + \int_\varepsilon^{\pi-\varepsilon} e^{-r\sin(\varepsilon)} dt \\ &= \frac{4}{r}(1 - e^{-\varepsilon r/2}) + (\pi - 2\varepsilon) e^{-r\sin(\varepsilon)}, \end{aligned}$$

which clearly tends to 0 when r tends to ∞.