Solutions to the final exam

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Problem 1 (35 points). Let $f(z) = \frac{2z-(3+i)}{(z-i)(z-3)}$.

(a) **(15 points).** Compute the Laurent series expansion of f in the annulus

$$A = \{1 < |z| < 3\}.$$

Solution: By splitting f into a sum of partial fractions, we obtain the following expression which holds for all $z \in A$.

$$\begin{aligned} f(z) &= \frac{1}{z-i} + \frac{1}{z-3} \\ &= \frac{1}{z} \cdot \frac{1}{1-\frac{i}{z}} + \frac{-1}{3} \cdot \frac{1}{1-\frac{1}{3}z} \\ &= \sum_{n=1}^{\infty} i^{n-1} z^{-n} + \sum_{n=0}^{\infty} \frac{-1}{3^{n+1}} z^n \end{aligned}$$

This is the desired Laurent series expansion.

(b) **(10 points).** Determine the radius of convergence for the power series expansion of f at the point $z_0 = 2 + i$.

Solution: Since the power series expansion of f at z_0 converges to f in any open disk centered at z_0 in which f is holomorphic, the radius of convergence is the distance from z_0 to the nearest singularity. One easily checks that 3 is the nearest singularity at a distance of $\sqrt{2}$ from z_0. Hence, the radius of convergence is $\sqrt{2}$.

Problem 2 (15 points). Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Show that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges when $|z| < 1$.

Solution: Suppose $|a_n| \leq M$ for all n. We may assume that $M > 0$. Letting R denote the radius of convergence, we have

$$1/R = \limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} M^{1/n} = 1,$$

from which it follows that $R \geq 1$. Hence, $\sum_{n=0}^{\infty} a_n z^n$ converges when $|z| < 1$.

Problem 3 (15 points). Let $f: \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Show that the function

$$M: [0, \infty) \to \mathbb{R}, \quad t \mapsto \max_{|z|=t} |f(z)|$$

is a strictly increasing function.

Solution: Let $t_1 < t_2$. We wish to show that $M(t_1) < M(t_2)$. By the maximum modulus principle, the maximum of $|f(z)|$ in the disk $\{|z| \leq t_2\}$ must occur on the boundary, i.e., we must have

$$|f(z)| < M(t_2), \quad \text{for } |z| < t_2.$$

In particular, this holds when $|z| = t_1$. Hence,

$$M(t_1) = \max_{|z|=t_1} |f(z)| < M(t_2).$$

Problem 4 (35 points). Let f be a complex-valued function that is holomorphic in an open subset of $\mathbb{C}$ containing the integers.

(a) **(10 points).** Show that the function $g(z) = f(z)\cot(\pi z)$ has an isolated singularity at each integer $n \in \mathbb{Z}$, and show that $\operatorname{res}_{z=n}(g) = \frac{f(n)}{\pi}$.

Solution: Let $n \in \mathbb{Z}$. Since $g(z) = \frac{f(z)\cos(\pi z)}{\sin(\pi z)}$, we see that n is an isolated singularity of g. By Exercise VIII.12.1, the residue is $\frac{f(n)\cos(\pi n)}{\pi \cos(\pi n)} = \frac{f(n)}{\pi}$.

For the rest of this problem, we let $f(z) = 1/(2z-1)^2$.

(b) **(12 points).** Show that $g(z) = f(z)\cot(\pi z)$ has a simple pole at $z = 1/2$ with residue $-\pi/4$.

Solution: Clearly $1/2$ is an isolated singularity of g. We have

$$(z-1/2)g(z) = \frac{z-1/2}{(2z-1)^2} \cdot \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{4} \cdot \frac{1}{\sin(\pi z)} \cdot \frac{\cos(\pi z) - \cos(\pi \cdot 1/2)}{z - 1/2},$$

which implies that

$$\lim_{z \to 1/2} (z-1/2)g(z) = \frac{1}{4} \cdot \frac{1}{\sin(\pi/2)} \cdot (-\pi \sin(\pi/2)) = -\pi/4.$$

Hence, $z = 1/2$ is a simple pole of g with residue $-\pi/4$.

For each natural number $N \in \mathbb{N}$, let γ_N^t, γ_N^b, γ_N^l, and γ_N^r be parametrizations of the top, bottom, left and right edges of the square S_N with vertices at $\pm(N+1/2) \pm (N+1/2)i$.

(c) **(18 points).** Show that each of the integrals $\int_{\gamma_N^t} g$, $\int_{\gamma_N^b} g$, $\int_{\gamma_N^l} g$, and $\int_{\gamma_N^r} g$ tends to 0 as N tends to infinity.
Hint: First show that $\cot(\pi z)$ is bounded on S_N by a constant independent of N. Then show that this implies the result.

Solution: As hinted, we first show that $\cot(\pi z)$ is bounded on S_N: We may parametrize the horizontal edges of S_N by $t \pm (N+1/2)i$, and the vertical edges by $\pm(N+1/2)+ti$. For the horizontal edges we have

$$\begin{aligned}\left|\frac{\cos\big(\pi(t\pm(N+1/2)i)\big)}{\sin\big(\pi(t\pm(N+1/2)i)\big)}\right| &= \left|\frac{ie^{\pi i(t\pm(N+1/2)i)}+ie^{-\pi i(t\pm(N+1/2)i)}}{e^{\pi i(t\pm(N+1/2)i)}-e^{-\pi i(t\pm(N+1/2)i)}}\right| \\ &\leq \left|\frac{e^{\mp\pi(N+1/2))}+e^{\pm\pi(N+1/2)}}{e^{\mp\pi(N+1/2))}-e^{\pm\pi(N+1/2)}}\right| \\ &= \frac{e^{\pi(N+1/2))}+e^{-\pi(N+1/2)}}{e^{\pi(N+1/2))}-e^{-\pi(N+1/2)}},\end{aligned}$$

which tends to 1 when N tends to ∞. Hence, $\cot(\pi z)$ is bounded on the horizontal edges. Similarly,

$$\begin{aligned}\left|\frac{\cos\big(\pi(\pm(N+1/2)+ti)\big)}{\sin\big(\pi(\pm(N+1/2)+ti)\big)}\right| &= \left|\frac{\cos(\pi/2+ti)}{\sin(\pi/2+ti)}\right| \\ &= \left|\frac{\sin(ti)}{\cos(ti)}\right| \\ &= \left|\frac{e^{-t}-e^{t}}{e^{-t}+e^{t}}\right| \\ &\leq 1,\end{aligned}$$

and it follows that $\cot(\pi z)$ is bounded on the vertical edges as well. Let M denote the bound. We have

$$\left|\int_{\gamma_N^t} g\right| = \left|\int_{\gamma_N^t} \frac{\cot(\pi z)}{(2z+1)^2}\right| \leq \frac{M}{(2N+2)^2} L(\gamma_N^t) \leq \frac{M}{2N+1},$$

which tends to zero when N tends to ∞. Together with an identical analysis of the other integrals, this yields the result.

(d) **(15 points).** Show that the series $\sum_{n=-\infty}^{\infty} \operatorname{res}_{z=n}(g)$ converges to $\pi/4$.

Solution: Since the only singularities of g are at the integers and at $z = 1/2$, it follows from the residue theorem that

$$\int_{S_N} g = 2\pi i \left(\operatorname{res}_{z=1/2}(g) + \sum_{n=-N}^{N} \operatorname{res}_{z=n}(g) \right).$$

Letting N tend to ∞, the left hand side tends to 0, and we obtain

$$\sum_{n=-\infty}^{\infty} \operatorname{res}_{z=n}(g) = -\operatorname{res}_{z=1/2}(g) = \pi/4.$$

(e) **(10 points).** Show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \pi^2/8$.

Solution: By (a), we have $\operatorname{res}_{z=n}(g) = \frac{1}{\pi(2n-1)^2}$. Hence, by (d), we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi(2n-1)^2} = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \pi/4,$$

and the result follows.

Problem 5 (10 points). Let $z_0 \in \mathbb{C}$ and let $f\colon \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function. Show that the derivative f' of f has an isolated singularity at z_0 with residue zero.

Solution: Let $\sum_{n=1}^{\infty} a_n(z-z_0)^{-n} + \sum_{n=0}^{\infty} b_n(z-z_0)^n$ be the Laurent series expansion of f at z_0. Since the convergence is uniform, we may differentiate term by term, so we have

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} -na_n(z-z_0)^{-n-1} + \sum_{n=0}^{\infty} nb_n(z-z_0)^{n-1} \\ &= \cdots + (-a_1)(z-z_0)^{-2} + b_1 + 2b_2(z-z_0) + \ldots, \end{aligned}$$

from which it follows that the residue is 0.

Problem 6 (35 points). Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk, and let $U = \{z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 1\}$.

(a) **(10 points).** Show that D and U are conformally equivalent.

Solution: Since U is simply connected, this follows immediately from the Riemann mapping theorem.

(b) **(10 points).** For each real number s consider the function

$$\phi_s \colon U \to U, \qquad z \mapsto z + is.$$

Show that ϕ_s is a conformal equivalence and that $\phi_{s+t} = \phi_s \circ \phi_t$ for all $s, t \in \mathbb{R}$.

Solution: Since ϕ_s is obviously holomorphic, and since any vertical translation of U is bijective, ϕ_s is a conformal equivalence. By definition, we have

$$\phi_{s+t}(z) = z+(s+t)i = z+ti+si = \phi_s(z+ti) = \phi_s(\phi_t(z)) = \phi_s \circ \phi_t(z).$$

Since this holds for each z, the result follows.

(c) **(15 points).** Let G be the set of conformal equivalences $D \to D$. Show that there exists an injective function $\Psi \colon \mathbb{R} \to G$ such that $\Psi(s+t) = \Psi(s) \circ \Psi(t)$ for all $s, t \in \mathbb{R}$ (in the language of algebra this means that the group G has a subgroup isomorphic to $\mathbb{R}$).

Solution: Let $f \colon D \to U$ be a conformal equivalence and consider the map

$$\Psi \colon \mathbb{R} \to G, \qquad s \mapsto f^{-1} \circ \phi_s \circ f.$$

This map satisfies

$$\Psi(s+t) = f^{-1} \circ \phi_{s+t} \circ f = f^{-1} \circ (\phi_s \circ \phi_t) \circ f = (f^{-1} \circ \phi_s \circ f) \circ (f^{-1} \circ \phi_t \circ f) = \Psi(s) \circ \Psi(t).$$

Since $\phi_s \neq \phi_t$ for $s \neq t$, it follows that Ψ is injective.

Problem 7 (20 points). Evaluate the integral $\int_0^\infty \frac{x^2}{1+x^4} dx$.

Solution: The singularities of the integrand in the upper half plane are $\pm\sqrt{2}/2 + i\sqrt{2}/2$. Since these are simple zeros of the denominator, it follows from Exercise VIII.12.1 that the residues are given by

$$\frac{\left(\pm \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2}{4\left(\pm \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^3} = \frac{1}{4}\left(\pm \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \pm\frac{\sqrt{2}}{8} - i\frac{\sqrt{2}}{8}.$$

Let $\gamma_R = Re^{it}$, $0 \le t \le \pi$. For z on γ_R, we have $\left|\frac{z^2}{1+z^4}\right| \le \frac{R^2}{R^4} = \frac{1}{R^2}$, which clearly tends to zero when R tends to ∞. By the residue theorem,

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = 2\pi i(-i\frac{\sqrt{2}}{4}) = \frac{\pi\sqrt{2}}{2},$$

and since the integrand is an even function, the desired integral equals $\frac{\pi\sqrt{2}}{4}$.
Problem 8 (15 points). Let $f(z) = \sin(z)$, and let ϕ_1 and ϕ_2 be the curves

$$\phi_1(t) = te^{\pi i/8}, \qquad \phi_2(t) = -i + e^{2\pi i(1/4-t)}.$$

Let $\gamma_1 = f \circ \phi_1$ and let $\gamma_2 = f \circ \phi_2$. Compute the angle between the tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$.

Solution: By conformality, the angle equals the angle between $\phi_1'(0)$ and $\phi_2'(0)$. Since ϕ_2 is a clockwise parametrized circle tangent to the real line at $\phi_2(0)$, $\phi'(0)$ is a positive real number (a computation shows that it is 2π). Since $\phi_1'(0) = e^{\pi i/8}$, the desired angle is $\pi/8$.