# Expected Value & Variance

| *Distribution* | *Description* | *Prob. Mass function* | $E[X]$ | Var$[X]$ |
|---|---|---|---|---|
| **Uniform** | $X$ takes each value in the range $R$ with equal probability. $R$ contains $n$ numbers. *Example*: Value shown on a fair die. | $f_X(k) = \begin{cases} \frac{1}{n} & k \in R \\ 0 & \text{else.} \end{cases}$ | $\frac{1}{n}\sum_{k\in R} k$ (average) | $\frac{1}{n}\sum_{k\in R}(k - E[X])^2$ |
| **Bernoulli** | A Bernoulli trial results in "success" with probability $p$ and "failure" with probability $1-p$. $X$ takes the value 1 in the event of success, and 0 for failure. | $f_X(k) = \begin{cases} p & k=1 \\ 1-p & k=0 \\ 0 & \text{else.} \end{cases}$ | $p$ | $p(1-p)$ |
| **Binomial** | A Bernoulli trial with probability $p$ of success is performed $n$ times. $X$ is the number of successes. | $f_X(k) = \begin{cases} \binom{n}{k}p^k(1-p)^k & {\scriptstyle 0\le k\le n \atop \text{integer}} \\ 0 & \text{else.} \end{cases}$ | $np$ | $np(1-p)$ |
| **Geometric** | A Bernoulli trial with probability $p$ of success is performed until the first success is achieved. $X$ is the number of failures before the first success. | $f_X(k) = \begin{cases} p(1-p)^k & {\scriptstyle k\ge 0 \atop \text{integer}} \\ 0 & \text{else.} \end{cases}$ | $\frac{1-p}{p}$ | $\frac{1-p}{p^2}$ |
| **Hypergeometric** | A jar contains $N$ marbles, of which $m$ are white and $N-m$ are black. A sample of $n$ marbles is drawn, and $X$ is the number of white marbles in the sample. | $f_X(k) = \begin{cases} \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} & {\scriptstyle 0\le k\le m \atop \text{integer}} \\ 0 & \text{else.} \end{cases}$ | $\frac{mn}{N}$ | $\frac{nm(N-m)(N-n)}{N^2(N-1)}$ (not in slides) |
| **Poisson** | $X$ is the number of times an event occurs, which is known to occur at an "average rate" $\lambda$, independently of the amount of time since the last event. | $f_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & {\scriptstyle k\ge 0 \atop \text{integer}} \\ 0 & \text{else.} \end{cases}$ | $\lambda$ | $\lambda$ |

## 1. Practice Problems

(1) Derive the formula for $E[X]$ for the Bernoulli random variable.

(2) Show that $E[X]$ for the Binomial and Hypergeometric random variables can be derived from the previous answer using the properties of Expected Value.

(3) Why is $\text{Var}[X]$ for the hypergeometric random variable not given by $\frac{mn(N-m)}{N^2}$?

(4) What is a fair price for the following games:
   (a) A fair coin is flipped until heads comes up, and the player wins a dollar for each tails she flipped before the heads.

   (b) The game runner and the player each roll a six-sided die. If the player has a strictly higher roll, then he wins a dollar.

   (c) The player watches a road which has red cars pass at a rate of 10 per hour. She wins a dollar for each red car she sees in the hour.

(5) Suppose I flip a penny, a nickel, a dime, and a quarter. Let $X$ be the value in cents of the coins showing heads. What is $E[X]$ and $\text{Var}[X]$?