(Problems selected from worksheets by Rob Bayer)

(1) **Direction Field Practice**. On the back of the page, there are 4 direction fields.

(a) Without thinking hardly at all, which one of these is for $y' = 1 + y$? Why?

(b) The differential equations for the other ones are $y' = x^2 - y^2, y' = y\sin(2x)$, and $y' = 1 - xy$. Determine which is which.

(c) Using the direction fields, sketch some solution curves to $y' = x^2 - y^2$.

Solution:

(a) The bottom right direction field. It is only dependent on y.

(b) $y' = x^2 - y^2$: Top left. $y' = y\sin(2x)$: top right. $y' = 1 - xy$: bottom left.

(c) Follow the arrows!

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(2) **Separable Equations word problems!**

(a) A tank initially contains 100L of water with 1000g of salt dissolved in it. Brine containing 50g/L of salt is pumped in at a rate of 2L/min. The solution is kept thoroughly mixed and solution leaves the tank at a rate of 2L/min. Set up and solve an initial value problem whose solution would give you the grams of salt in the tank at time t.
Hint 1: The rate of change of the amount of salt is the same as (the amount of salt coming in) $-$ (the amount of salt leaving).
Hint 2: The amount of salt leaving depends on how much salt is in the solution now.

Solution: The differential equation modeling this situation is:

$$y' = 50\ g/L \cdot (2L/min) - \frac{y}{100\ L} \cdot (2L/min)$$

We ignore the units for now, and solve:

$$\frac{dy}{dt} = 100 - \frac{y}{50}$$

Using separable equations, this turns into:

$$50 \int \frac{dy}{5000 - y} = \int dt \Rightarrow -50 \ln|5000 - y| = t + C \Rightarrow y = 5000 - Ae^{-t/50}$$

where A is an arbitrary nonzero constant. Bringing the units back, we have:

$$y = \left(5000 - Ae^{-t/50\text{ min}}\right) \text{ grams}$$

Checking $A = 0$, we find that the constant function $y = 5000$ grams is an equilibrium solution, so we let A take any value.
Now we want to find the solution with the given initial condition $y(0) = 1000$ grams. This gives 1000 grams $= 5000 - A$ grams. So $A = 4000$ and $y = 5000 - 4000e^{-t/50\text{ min}}$ grams.

(b) A certain curve in the plane has the property that every normal line (that is, a line perpendicular to the tangent line) to the curve passes through $(2, 0)$. Find the equation for this curve if you know it passes through $(1, 1)$.
Hint: What this problem is really asking you is to find a curve where at each point (x, y), the tangent line (which has slope dy/dx) is perpendicular to the line from $(2, 0)$ to (x, y) (what is the slope of this line?).

Solution: The line from (x, y) to $(2, 0)$ has slope $\frac{y}{x-2}$ (rise over run!). We want the derivative to be perpendicular to this, so we set it equal to the negative reciprocal:

$$\frac{dy}{dx} = -\frac{x-2}{y} \Rightarrow \int y dy = \int 2 - x dx \Rightarrow \frac{1}{2}y^2 = 2x - \frac{1}{2}x^2 + C.$$

Plugging in the "initial value," i.e. $(1, 1)$, we obtain $1/2 = 2 - 1/2 + C \Rightarrow C = -1$. So the equation defining the desired curve is $y^2 = 4x - x^2 - 2$ (multiplying the whole thing by 2 for convenience).

This can also be written as $y^2 + x^2 - 4x + 4 = 2 \Leftrightarrow y^2 + (x-2)^2 = (\sqrt{2})^2$: the equation for a circle of radius $\sqrt{2}$ centered at $(2, 0)$.

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(3) Consider the differential equation $y' = (y-3)(y+2)^2(y+4)$.

(a) Without solving for y, what are the equilibrium solutions of this differential equation?

(b) Sketch a graph with the equilibrium solutions, and other solutions in between. (Consider where the slope is positive or negative.)

(c) Use separable equations to find an expression for x in terms of y. (y can't be written simply as a function of x.)

Solution:

(1) We want to plug in a constant function y, such that y' as calculated by the differential equation will be zero. This works for:

$$y = 3, y = -2, y = -4$$

(3)

$$x = \frac{1}{700}\left(\frac{70}{y+2} + 4\log|y-3| + 21\log|y+2| - 25\log|y+4|\right) + C.$$
